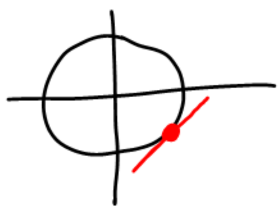
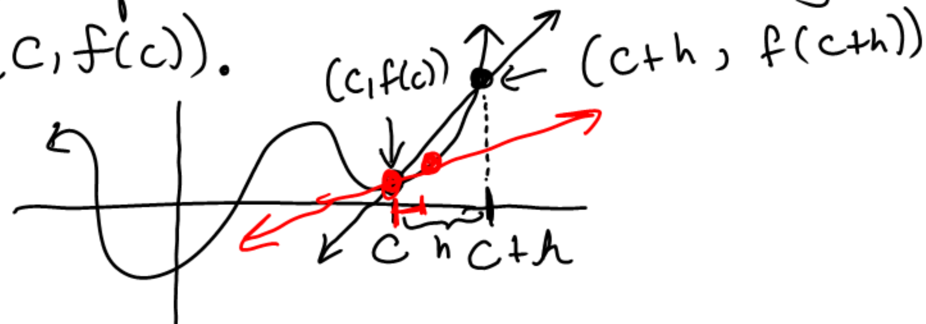
GOAL: Become familiar with the limit definition of the derivative. Be able to solve the Tangent Problem for various functions.

We say a line is tangent to an object if it touches it only once.



The Tangent Problem Given a function f and a point c, determine the equation of the line tangent to f at the point $(c, f(c))$.



We can approximate a solution a secant line
The slope of the secant line (difference quotient)

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(c+h) - f(c)}{c+h-c} = \frac{f(c+h) - f(c)}{h} \approx \text{Slope of the tangent line}$$

Notice: We get a better approximation by moving closer to c (make h smaller)

Def The derivative of f at $x=c$ is the quantity

$$\underbrace{f'(c)}_{\text{"f prime of c"}} \stackrel{\text{def}}{=} \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

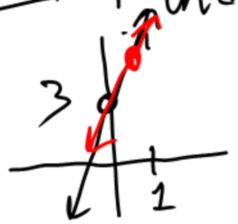
NOTE: ① Some textbooks use ΔX as opposed to h

② If $f'(c)$ exists, the line with slope $f'(c)$ at the point c.

$$y - f(c) = f'(c)(x - c)$$

is the solution to the textbook.

Ex1 Find $f'(1)$ if $f(x) = 2x+3$



$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{2(1+h)+3-[2(1)+3]}{h}$$

$$= \lim_{h\to 0} \frac{2+2h+3-5}{h} = \lim_{h\to 0} \frac{2h}{h} = 2$$

Derivative As A Function

A point x $\longrightarrow$ $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ $\longrightarrow$ The slope of the tangent line at the point x

Example 2 Find $f'(x)$ if $f(x) = x^2$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h\to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h\to 0} [2x + h] = 2x$$

Example 2(b) Find the equation of the line tangent to x^2 at $x=-1$.

Slope : $f'(-1) = 2(-1) = -2$

Point : $(-1, f(-1)) = (-1, (-1)^2) = (-1, 1)$

$$y - 1 = -2(x - (-1))$$

$$y - 1 = -2x - 2 \longrightarrow y = -2x - 1$$

Def If $f'(c)$ exists, we say f is <u>differentiable</u> at c.

NOTATION: $y = f(x)$

Derivative	Evaluate @ c
$f'(x)$	$f'(c)$
y'	$y'(c)$
$\frac{dy}{dx}$	$\left.\frac{dy}{dx}\right\vert_{x=c} = \left.\frac{dy}{dx}\right]_{x=c}$
$\frac{df}{dx}$	$\left.\frac{df}{dx}\right\vert_{x=c}$

Example 3 Find $\frac{d}{dx}(x^3)$ Google "Pascal's Triangle"

$$\frac{d}{dx}(x^3) = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \lim_{h\to 0} \left[3x^2 + 3xh + h^2\right] = 3x^2$$

1
1 1
1 2 1
1 3 3 1

Ex 4 If $y = \sqrt{x}$ $(x>0)$, find y' $[a^2 - b^2 = (a+b)(a-b)]$

$$y' = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}\left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right) = \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}}$$

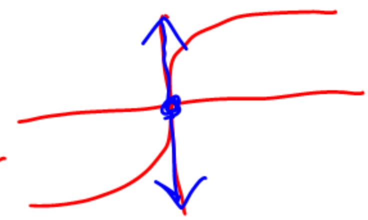
$$= \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Question: When does differentiability fail?

Answer 1 A vertical tangent line

Ex 5 When is $\sqrt[3]{x}$ differentiable?

Take for granted $(\sqrt[3]{x})' = \frac{1}{3\sqrt[3]{x^2}}$



Anywhere but 0

Answer 2: Discontinuity

Theorem ① If f is differentiable at c, f is continuous at c

Ie, if f is discontinuous at c, it is not differentiable.

Ex 6 When is $\frac{d}{dx}\left(\frac{1}{x}\right)$ well defined?

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{\frac{x}{(x+h)x} - \frac{x+h}{x(x+h)}}{h}$$

$$= \lim_{h\to 0} \left[\frac{x-x-h}{x(x+h)}\right] \cdot \frac{1}{h} = \lim_{h\to 0} \frac{-h}{x(x+h)} \cdot \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}$$

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \begin{cases} -\frac{1}{x^2} & x \neq 0 \\ \text{undefined} & x = 0 \end{cases}$$

Answer 3 A corner or kink

Ex $|x|$ is not differentiable at 0