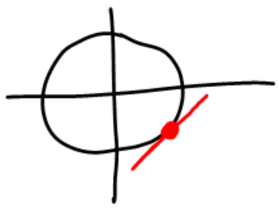
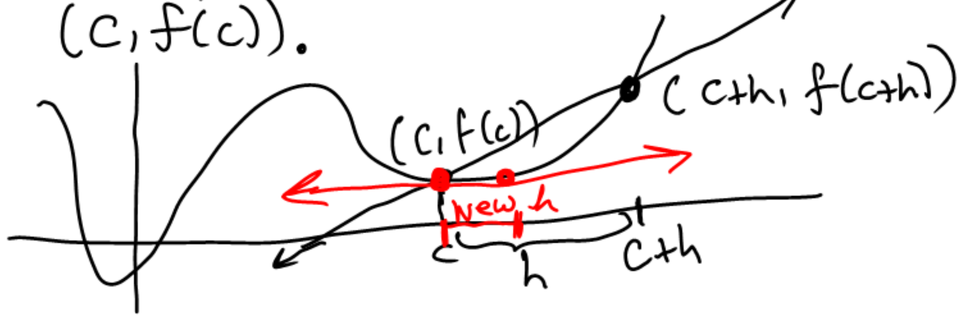
GOAL: Become familiar with the limit definition of the derivative. Be able to solve the Tangent Problem for various functions.

We say a line is <u>tangent</u> to an object if it touches it only once.



<u>The Tangent Problem</u> Given a function f and a point c, determine the equation of the line tangent to f at the point $(c, f(c))$.



We can approximate the solution using a secant line. The slope of the secant line (difference quotient)

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(c+h) - f(c)}{(c+h) - c} = \frac{f(c+h) - f(c)}{h} \approx \text{Slope of the tangent line}$$

<u>NOTE</u>: We get a better answer by moving closer to c (make h smaller).

<u>Def</u> The <u>derivative</u> of f at $x = c$ is defined by

$$\underbrace{f'(c)}_{\text{"f prime of c"}} \overset{\text{def}}{=} \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

If $f'(c)$ exists, we say f is <u>differentiable</u> at c.

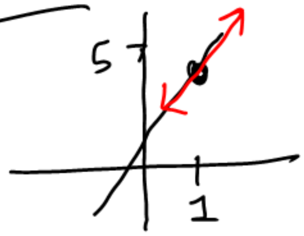
NOTE: ① Some textbooks use Δx as opposed to h.

② If $f'(c)$ exists, then the line with slope $f'(c)$ containing the point $(c, f(c))$

$$y - f(c) = f'(c)(x - c)$$

is the solution to the Tangent Problem.

Ex 1 Find $f'(1)$ if $f(x) = 2x + 3$



$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{2(1+h) + 3 - [2(1) + 3]}{h}$$

$$= \lim_{h \to 0} \frac{2 + 2h + 3 - 2 - 3}{h} = \lim_{h \to 0} \frac{2h}{h} = 2$$

Derivative As A Function

A point x $\longrightarrow$ $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ $\longrightarrow$ The slope of the tangent line at the point x

Ex 2 If $f(x) = x^2$, find $f'(x)$

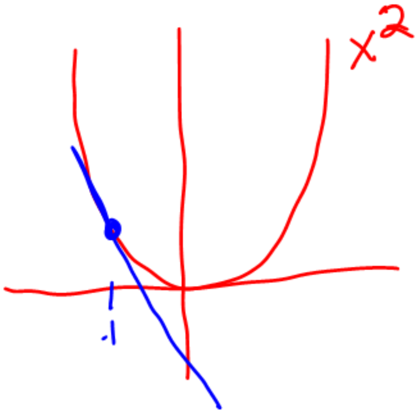
$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} [2x + h] = 2x$$

Ex 2b Find the equation of the line tangent to x^2 at $x = -1$

Slope: $f'(-1) = 2(-1) = -2$

Point: $(-1, f(-1)) = (-1, (-1)^2) = (-1, 1)$



$$y - 1 = -2(x - (-1))$$
$$y - 1 = -2(x+1)$$
$$y - 1 = -2x - 2$$
$$y = -2x - 1$$

NOTATION: $y = f(x)$

Derivative	Evaluate at c
$f'(x)$	$f'(c)$
y'	$y'(c) = y'\Big\vert_{x=c}$
$\dfrac{df}{dx}$	$\dfrac{df}{dx}\Big\vert_{x=c}$
$\dfrac{dy}{dx}$	$\dfrac{dy}{dx}\Big\vert_{x=c}$

__Ex 3__ $\frac{d}{dx}(x^3)$ Google Pascal's Triangle

$$\frac{d}{dx}(x^3) = \lim_{h\to 0}\frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0}\frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

$$= \lim_{h\to 0}\frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \lim_{h\to 0}\left[3x^2 + 3xh + h^2\right] = 3x^2$$

1
1 1
1 2 1
1 3 3 1

__Example 4__ Find y' if $y = \sqrt{x}$ $(x>0)$ $\left[a^2 - b^2 = (a-b)(a+b)\right]$

$$y' = \lim_{h\to 0}\frac{\sqrt{x+h} - \sqrt{x}}{h}\left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right)$$

$$= \lim_{h \to 0} \frac{(x+h) - (x)}{h\left[\sqrt{x+h} + \sqrt{x}\right]} \qquad \frac{\sqrt{2}}{2} \qquad \frac{1}{\sqrt{2}}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}\left[\sqrt{x+h} + \sqrt{x}\right]} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

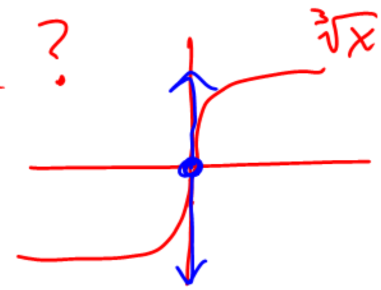
$$= \frac{1}{2\sqrt{x}}$$

Question: When does differentiability fail?

Answer 1 A vertical tangent line

Example 5 When is $\sqrt[3]{x}$ differentiable?

Take for granted $\frac{d}{dx}\left(\sqrt[3]{x}\right) = \frac{1}{3\sqrt[3]{x^2}}$



Answer 2 Discontinuity

Theorem ① If f is differentiable at c, f is continuous at c.

② If f is discontinuous at c, then it is not differentiable at c.

Example 6 $\frac{d}{dx}\left(\frac{1}{x}\right)$

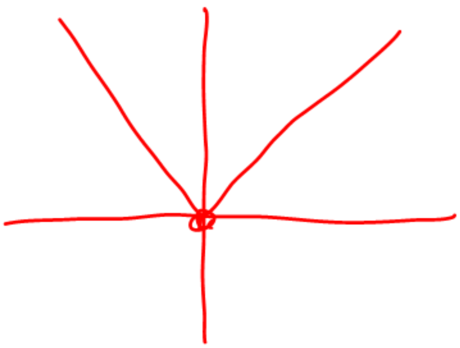
$$\lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{1}{h}\left[\frac{x}{(x+h)x} - \frac{x+h}{x(x+h)}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{x - x - h}{x(x+h)}\right] = \lim_{h \to 0} \frac{1}{h}\left[\frac{-h}{x(x+h)}\right] = \frac{-1}{x^2}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{x}\right) = \begin{cases} -\frac{1}{x^2} & x \neq 0 \\ \text{undefined} & x = 0 \end{cases}$$

Answer 3 Any kind of corner or kink

Ex 7 $|x|$ is not differentrable at 0.



This is an example of a fcn that is cont. at 0, but not diff at 0.