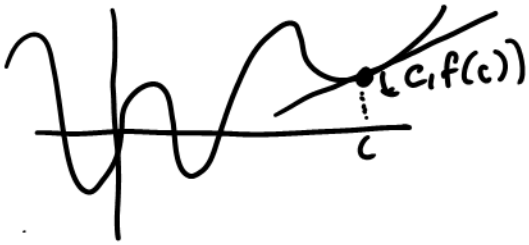
Lecture 5 The Derivative

We say a line is <u>tangent</u> to an object if it touches the object only once.



<u>The Tangent Problem</u> Given a graph f and a point c, determine the equation of the tangent line (tangent to f) at point

We can approximate a solution using a secant line. The slope of the secant line (<u>difference quotient</u>)

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(c+\Delta x) - f(c)}{(c+\Delta x) - c} = \frac{f(c+\Delta x) - f(c)}{\Delta x} \approx \text{Slope of the tangent line}$$

<u>Notice</u>: We get a better approximation by moving closer to c (make Δx smaller)

<u>Def</u> The <u>derivative</u> of f at $x=c$ is the quantity

$$f'(c) \overset{\text{def}}{=} \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x}$$

If $f'(c)$ exists, we say f is <u>differentiable</u> at $x=c$

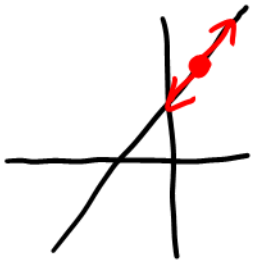
NOTE: ① Some textbooks use h instead of Δx

② If $f'(c)$ exists, the line with slope $f'(c)$ containing the point $(c, f(c))$

$$y - f(c) = f'(c)(x-c)$$

is the solution to the tangent problem.

Ex/ Find the equation of the tangent line of $f(x)=2x+3$ at $x=1$.



Sol/ Find Slope:

$$f'(1)=\lim_{\Delta x\to 0}\frac{f(1+\Delta x)-f(1)}{\Delta x}=\lim_{\Delta x\to 0}\frac{2(1+\Delta x)+3-(2(1)+3)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{2+2\Delta x+3-5}{\Delta x}=\lim_{\Delta x\to 0}\frac{2\Delta x}{\Delta x}=\lim_{\Delta x\to 0}2=2$$

Point: $(1, f(1))=(1,5)$

Equation: $y-f(1)=f'(1)(x-1)$

$y-5=3(x-1)\longrightarrow y=3x+2$

The Derivative As a Function

A point x → $f'(x)=\lim_{\Delta x\to 0}\frac{f(x+h)-f(x)}{h}$ → The slope of the tangent line at x

NOTATION:

Function	Evaluating at c
$f'(x)$	$f'(c)$
y'	$y'(c)$
$\frac{df}{dx}$	$\left.\frac{df}{dx}\right\|_{x=c}=\left.\frac{df}{dx}\right]_{x=c}$
$\frac{dy}{dx}$	$\left.\frac{dy}{dx}\right\|_{x=c}$

Ex/ If $f(x)=x^2$, find $f'(x)$

$$f'(x)=\lim_{\Delta x\to 0}\frac{(x+\Delta x)^2-x^2}{\Delta x}=\lim_{\Delta x\to 0}\frac{x^2+2x\Delta x+(\Delta x)^2-x^2}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{2x\Delta x+(\Delta x)^2}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta x[2x+\Delta x]}{\Delta x}=\lim_{\Delta x\to 0}(2x+\Delta x)=2x$$

Ex/ Find $\frac{d}{dx}(x^3)$ Google "Pascal's Triangle"

$$\frac{d}{dx}(x^3) = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^3 - x^3}{\Delta x} = \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x[3x^2 + 3x\Delta x + (\Delta x)^2]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (3x^2 + 3x\Delta x + (\Delta x)^2) = 3x^2$$

1
1 1
1 2 1
1 3 3 1

$a^2 - b^2 = (a+b)(a-b)$

Ex/ If $y = \sqrt{x}$ $(x>0)$, find y'

$$y' = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \left(\frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}}\right)$$

$$= \lim_{\Delta x \to 0} \frac{x+\Delta x - x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})}$$

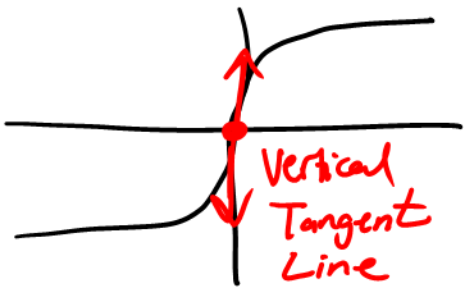
$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}} \xrightarrow{\text{When } x>0} \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Question When does differentiability fail?

Answer 1 A vertical tangent line

Ex/ When is $y = \sqrt[3]{x}$ differentiable?

Take for granted $y' = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$



Answer 2 Discontinuity

Theorem ① If f is differentiable at a point c, f is continuous at c.

② If f is not continuous at c, it is not differentiable at c

Ex/ When is $\lim_{\Delta x \to 0} \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x}$ well defined?

$$\lim_{\Delta x \to 0} \frac{x - x - \Delta x}{x(x+\Delta x)} \cdot \frac{1}{\Delta x} = \lim_{\Delta x \to 0} \frac{-\Delta x}{(\Delta x) \times (x+\Delta x)} = -\frac{1}{x^2} = \frac{d}{dx}\left(\frac{1}{x}\right)$$

$\frac{1}{x}$ is discontinuous at 0, so

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \begin{cases} \text{undefined} & x=0 \\ -\frac{1}{x^2} & x\neq 0 \end{cases}$$

Answer 3 There is a "corner" or "kink" in the graph

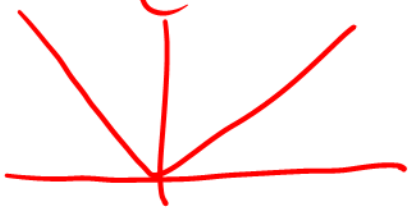
Ex/ When is $y=|x|$ differentiable? We claim $|x|$ is not differentiable at 0

$$y' = \lim_{\Delta x\to 0} \frac{|0+\Delta x| - |0|}{\Delta x}$$

$$\lim_{\Delta x\to 0^+} \frac{|0+\Delta x| - |0|}{\Delta x} = \lim_{\Delta x\to 0^+} \frac{\Delta x}{\Delta x} = 1$$

$$\lim_{\Delta x\to 0^-} \frac{|0+\Delta x| - |0|}{\Delta x} = \lim_{\Delta x\to 0^-} \frac{-\Delta x}{\Delta x} = -1$$

$$|x| = \begin{cases} x & x\geq 0 \\ -x & x<0 \end{cases}$$



$$y' = \begin{cases} 1 \text{ if } x>0 \\ -1 \text{ if } x<0 \\ \text{undefined if } x=0 \end{cases} \overset{\text{def}}{=} \operatorname{sgn}(x)$$

sign

$|x|$ is an example of a function that is continuous at 0, but not differentiable at 0.