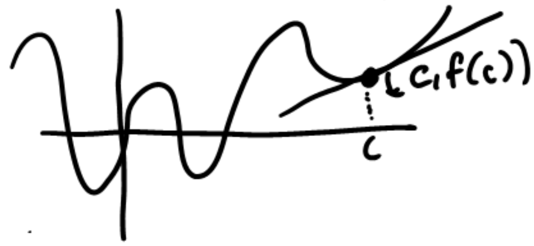
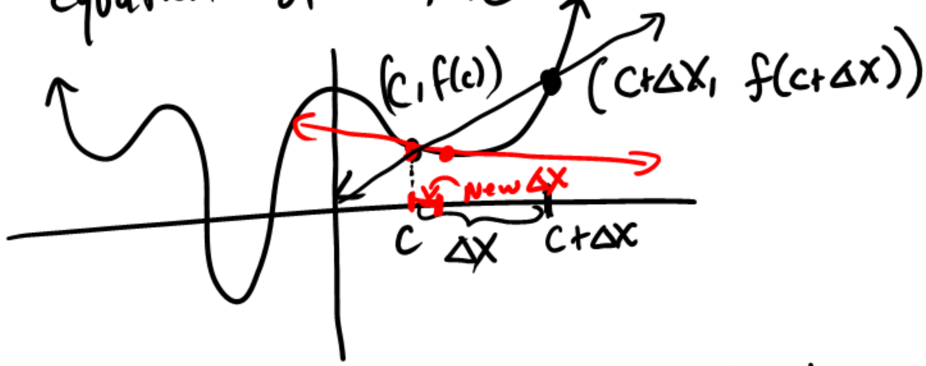
Lecture 5 The Derivative

We say a line is <u>tangent</u> to an object if it touches the object only once.



<u>The Tangent Problem</u> Given a function f and a point c, determine the equation of the line tangent to f at point c.



We approximate it by using a secant line.

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(c+\Delta x) - f(c)}{c + \Delta x - c} = \frac{f(c+\Delta x) - f(c)}{\Delta x} \approx \begin{array}{l}\text{The slope}\\ \text{of the}\\ \text{tangent line}\end{array}$$

<u>Notice:</u> We can get better approximation by moving closer to c (make Δx small)

Def The <u>derivative</u> of f at $x = c$ is the quantity

$$\underbrace{f'(c)}_{\text{"f prime of c"}} \stackrel{\text{def}}{=} \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x}$$

If $f'(c)$ exists, we say f is <u>differentiable</u> at c.

<u>NOTE</u> ① Some textbooks use h instead of Δx.

② If $f'(c)$ exists, the line with slope $f'(c)$ containing the point $(c, f(c))$

$$y - f(c) = f'(c)(x - c)$$

is the solution to the Tangent Problem.

Ex/ Find the equation of the tangent line of $f(x) = 2x+3$ at $x=1$.

Slope: $f'(1) = \lim_{\Delta x \to 0} \frac{f(1+\Delta x) - f(1)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\overbrace{2(1+\Delta x)+3}^{} - \overbrace{(2(1)+3)}^{}}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{2+2\Delta x+3-5}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\Delta x}{\Delta x} = \lim_{\Delta x \to 0} 2 = 2$

Point: $(1, f(1)) = (1,5)$

Equation: $y - f(1) = f'(1)(x-1)$

$y - 5 = 2(x-1) \longrightarrow y = 2x+3$

The Derivative As A Function

A point x $\longrightarrow$ $\boxed{f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}}$ $\longrightarrow$ The slope of the tangent line at x

NOTATION:

Function	Evaluating at $x=c$
$f'(x)$	$f'(c)$
y'	$y'(c)$
$\frac{df}{dx} = \frac{d}{dx}(f(x))$	$\frac{df}{dx}\Big\vert_{x=c} = \frac{df}{dx}\Big]_{x=c}$
$\frac{dy}{dx}$	$\frac{dy}{dx}\Big\vert_{x=c}$

Note The quantity $\frac{f(x+\Delta x) - f(x)}{\Delta x}$ is called the difference quotient

$x^2 + \Delta x$ ✗ common mistake $f(x)+\Delta x$ $f(x+\Delta x)$

Ex/ If $f(x) = x^2$, find $f'(x)$

$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{\Delta x[2x+\Delta x]}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x) = 2x$$

Ex/ Compute $\frac{d}{dx}(x^3)$ Google "Pascal's Triangle"

1
0 1 1 0
1 2 1
1 3 3 1

$$\frac{d}{dx}(x^3) = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^3 - x^3}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x} = \lim_{\Delta x \to 0} \left(3x^2 + 3x\Delta x + (\Delta x)^2\right)$$

$$= 3x^2$$

$a^2 - b^2 = (a+b)(a-b)$

Ex/ If $y = \sqrt{x}$ $(x > 0)$, find y'

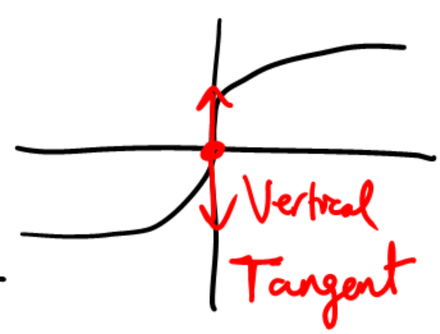
$$y' = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \left(\frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}}\right) = \lim_{\Delta x \to 0} \frac{x + \Delta x - x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})} = \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}}$$

When $x > 0$ $\quad \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$

Question When does differentiability fail?

Answer 1 A vertical tangent line



Ex/ When is $y = \sqrt[3]{x}$ differentiable?

Take for granted $y' = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$

Answer 2 A discontinuity

Theorem ① If f is differentiable at $x = c$, f is continuous at $x = c$

② If f is discontinuous at c, it is not differentiable at c

Ex/ When is $f(x) = \frac{1}{x}$ differentiable?

$$f'(x) = \lim_{\Delta x \to 0} \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x} = \lim_{\Delta x \to 0} \frac{x - x - \Delta x}{x(x+\Delta x)} \cdot \frac{1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-\cancel{\Delta x}}{(\cancel{\Delta x})\, x(x+\Delta x)} = \lim_{\Delta x \to 0} \frac{-1}{x(x+\Delta x)} = -\frac{1}{x^2}$$

$$f'(x) = \begin{cases} \text{undefined} & x = 0 \\ -\frac{1}{x^2} & x \neq 0 \end{cases}$$

<u>Answer 3</u> There is a "corner" or "kink" in the graph

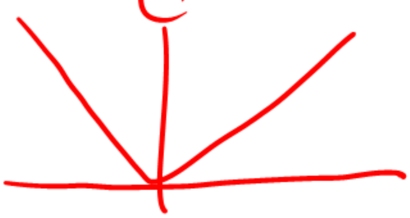
Ex/ When is $y = |x|$ differentiable? We claim $|x|$ is not differentiable at 0

$$y'\Big|_{x=0} = \lim_{\Delta x \to 0} \frac{|0+\Delta x| - |0|}{\Delta x}$$

$$\lim_{\Delta x \to 0^+} \frac{|0+\Delta x| - |0|}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{\Delta x}{\Delta x} = 1$$

$$\lim_{\Delta x \to 0^-} \frac{|0+\Delta x| - |0|}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{-\Delta x}{\Delta x} = -1$$

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$



$$y' = \begin{cases} 1 \text{ if } & x > 0 \\ -1 \text{ if } & x < 0 \\ \text{undefined if } & x = 0 \end{cases} \overset{\text{def}}{=} \operatorname{sgn}(x)$$

$|x|$ is an example of a function that is continuous at 0, but not differentiable at 0.