GOAL: Compute derivatives without relying on the definition.

Summary:

$\frac{d}{dx}(x^p) = px^{p-1}$	$\frac{d}{dx}(e^x) = e^x$
$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\cos x) = -\sin x$
$\frac{d}{dx}(f(x) \pm g(x)) = \frac{df}{dx} + \frac{dg}{dx}$	$\frac{d}{dx}(k \cdot f(x)) = k\frac{df}{dx}$

Polynomials and Power Functions

Function	Degree	Derivative
Constant	0	0
x	1	$1 = 1x^{1-1}$
x^2	2	$2x = 2x^{2-1}$
x^3	3	$3x^2 = 3x^{3-1}$
x^4	4	$4x^3 = 4x^{4-1}$

Constan c.

$$(c') = \lim_{h\to 0} \frac{c-c}{h} = 0$$

$$(x)' = \lim_{h\to 0} \frac{(x+h)-x}{h} = 1$$

Theorem (Power Rule) Let p be any real number. Then

$$\frac{d}{dx}(x^p) = p\,x^{p-1} = px^{p-1}$$

Why? We will show this in steps over several lectures.

Ex 1

① $\frac{d}{dx}(x^{10}) = 10 \cdot x^{10-1} = 10x^9$ ($p=10$)

② $\frac{d}{dx}\left(\frac{1}{x^2}\right) = \frac{d}{dx}(x^{-2}) = -2 \cdot x^{-2-1} = -2x^{-3} = \frac{-2}{x^3}$ ($p=-2$)

③ $\frac{d}{dx}\left(\sqrt[3]{x}\right) = \frac{d}{dx}\left(x^{\frac{1}{3}}\right) = \frac{1}{3}\cdot x^{\frac{1}{3}-1} = \frac{1}{3}x^{-\frac{2}{3}}$

$p = \frac{1}{3}$

$= \frac{1}{3\sqrt[3]{x^2}}$

Question: How can we differentiate $x^2 - 2x + 3$?

Theorem [Linearity] Let f and g be differentiable fcns

① Sum/Difference Rule

$$[f(x) \pm g(x)]' = f'(x) \pm g'(x)$$

② Constant-Multiple Rule:

$$[k\cdot f(x)]' = k\cdot f'(x) \text{ for } k \text{ constant}$$

Ex2

$$(x^2 - 2x + 3)' = (x^2)' - (2x)' + (3)'$$
$$= (x^2)' - 2(x)' + (3)'$$
$$= 2\cdot x^{2-1} - 2\cdot 1 + 0$$
$$= 2x - 2$$

Ex3 (HW 6, #8)

$$\left[\frac{(2x)^{\pi} - x^{3.5}}{\sqrt{x}}\right]' = \left[\frac{2^{\pi}\cdot x^{\pi} - x^{3.5}}{x^{0.5}}\right]' = \left[\frac{2^{\pi}x^{\pi}}{x^{0.5}} - \frac{x^{3.5}}{x^{0.5}}\right]'$$

$$= \left[2^{\pi}x^{\pi-0.5} - x^3\right]'$$

$$= \left[2^{\pi}x^{\pi-0.5}\right]' - \left[x^3\right]'$$

$= 2^{\pi} \cdot (\pi - 0.5) x^{\pi - 1.5} - 3x^2$

Sine and Cosine

Theorem ① $\frac{d}{dx}(\sin x) = \cos x$

② $\frac{d}{dx}(\cos x) = -\sin x$ Don't forget

Why? ① Textbook

Recall $\cos(a+b) = \cos a \cos b - \sin a \sin b$

② $\frac{d}{dx}(\cos x) = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$

$= \lim_{h \to 0} \frac{\cos x \cosh - \sin x \sinh - \cos x}{h}$

$= \lim_{h \to 0} \left[\cos x \underbrace{\left(\frac{\cos h - 1}{h} \right)}_{\to 0} - \sin x \underbrace{\left(\frac{\sin h}{h} \right)}_{\to 1} \right] = -\sin x$

Ex 4 If $y = 2\sin x - 5\cos x$, find y'

$y' = 2(\sin x)' - 5(\cos x)'$

$= 2\cos x - 5(-\sin x)$

$= 2\cos x + 5\sin x$

Ex 5 Find the equation of the tangent line for $f(x) = 3\sin x$ at $x = \pi$

Slope: Need $f'(\pi)$

$f'(x) = 3\cos x$

$f'(\pi) = 3\cos\pi = 3(-1) = -3$

Point: $(\pi, f(\pi)) = (\pi, \sin\pi) = (\pi, 0)$

$\longrightarrow y - f(\pi) = f'(\pi)(x-\pi)$

$$y - 0 = -3(x - \pi)$$

$$y = -3x + 3\pi$$

$\underline{e^x:}$

$$\frac{d}{dx}\left(e^x\right) = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x e^h - e^x}{h}$$

$$= \lim_{h\to 0} e^x \underbrace{\left[\frac{e^h - 1}{h}\right]}_{\to 1} = e^x$$

<u>Theorem</u> $y = e^x$ is the only function that satisfies the following Initial Value Problem (IVP)

$$\begin{cases} \frac{dy}{dx} = y & \leftarrow \frac{d}{dx}(e^x) = e^x \\ y(0) = 1 & \leftarrow e^0 = 1 \end{cases}$$

<u>Ex</u> What is $y'(0)$ if $y = 5e^x + 7$

$$y' = 5\left[e^x\right]' + (7)' = 5e^x$$

$$y'(0) = 5 \cdot e^0 = \boxed{5}$$

<u>Ex</u> When does $\frac{dy}{dx} = 1$ if $y = 72e^x$

$$\frac{dy}{dx} = [72e^x]' = 72[e^x]'$$

$$= 72e^x \overset{\text{set}}{=} 1$$

$$e^x = \frac{1}{72}$$

$$\ln(e^x) = \ln\left(\frac{1}{72}\right)$$

$$x = \ln\left(\frac{1}{72}\right)$$

$$x = \ln\left(72^{-1}\right)$$

$$\boxed{x = -\ln 72}$$

Ex HW 6, #9

Find the equation for the tangent line of $f(x) = \frac{x^4}{8} - \frac{16}{x^2}$ at $x=2$

$$f(x) = \frac{1}{8}x^{4} - 16x^{-2} \quad {}^{-1} \quad {}^{-1}$$

$$f'(x) = \frac{1}{8} \cdot 4x^3 - 16\left[-2x^{-3}\right]$$

$$= \frac{1}{2}x^3 + 32 \cdot \frac{1}{x^3} = \underbrace{\frac{1}{2} \cdot 8 + \frac{32}{8}}_{\text{Plugging in 2}}$$

$$f'(2) = 4 + 4 = 8$$

$$f(2) = \frac{2^4}{8} - \frac{16}{2^2} = 2 - 4 = -2$$

$$y - (-2) = 8(x-2)$$

$$y + 2 = 8x - 16$$

$$y = 8x - 18$$