GOAL: Compute derivatives without relying on the definition.

Summary:

$\frac{d}{dx}(x^p) = px^{p-1}$	$\frac{d}{dx}(e^x) = e^x$
$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\cos x) = -\sin x$
$\frac{d}{dx}(f(x) \pm g(x)) = \frac{df}{dx} + \frac{dg}{dx}$	$\frac{d}{dx}(k \cdot f(x)) = k\frac{df}{dx}$

Polynomials and Power Functions

Function	Degree	Derivative
Constant	0	0
x	1	$1 = 1 \cdot x^{1-1}$
x^2	2	$2x = 2x^{2-1}$
x^3	3	$3x^2 = 3x^{3-1}$
x^4	4	$4x^3 = 4x^{4-1}$

Constan c:

$$(c)' = \lim_{h\to 0} \frac{c-c}{h} = 0$$

$$(x)' = \lim_{h\to 0} \frac{(x+h)-x}{h} = 1$$

Theorem (Power Rule) Let p be any real number. Then

$$\frac{d}{dx}(x^p) = p \cdot x^{p-1} = p x^{p-1}$$

Why? We will show in steps over several lectures.

Ex 1

① $\frac{d}{dx}(x^{10}) = 10 \cdot x^{10-1} = 10x^9$ $\quad p = 10$

② $\frac{d}{dx}\left(\frac{1}{x^2}\right) = \frac{d}{dx}\left(x^{-2}\right) = -2 \cdot x^{-2-1} = -2x^{-3} = \frac{-2}{x^3}$ $\quad p = -2$

③ $\frac{d}{dx}\left(\sqrt[3]{x}\right) = \frac{d}{dx}\left(x^{\frac{1}{3}}\right) = \frac{1}{3}\cdot x^{\frac{1}{3}-1} = \frac{1}{3}x^{-\frac{2}{3}}$

$\uparrow$ $p=\frac{1}{3}$ $\qquad \frac{1}{3}-\frac{3}{3}$

$x^{\frac{p}{q}} = \sqrt[q]{x^p} = \left(\sqrt[q]{x}\right)^p$ $\qquad = \frac{1}{3\sqrt[3]{x^2}}$

Question: How can we differentiate x^2-2x+3?

Theorem [Linearity] Let f and g be differentiable fcns

① Sum/Difference Rule

$$[f(x) \pm g(x)]' = f'(x) \pm g'(x)$$

② Constant-Multiple Rule

$$[k\cdot f(x)]' = k\,[f(x)]' \text{ for constant } k$$

Ex2 $[x^2-2x+3]' = [x^2]' - [2x]' + [3]'$

$= [x^2]' - 2[x]' + [3]'$

$= 2\cdot x^{2-1} - 2\cdot 1x^{1-1} + 0$

$= 2x - 2$

Ex3 (HW6, #8)

$$\left[\frac{(2x)^{\pi} - x^{3.5}}{\sqrt{x}}\right]' = \left[\frac{2^{\pi}x^{\pi} - x^{3.5}}{x^{0.5}}\right]' = \left[\frac{2^{\pi}x^{\pi}}{x^{0.5}} - \frac{x^{3.5}}{x^{0.5}}\right]'$$

$$= \left[2^{\pi}x^{\pi-0.5} - x^{3}\right]'$$

$$= 2^{\pi}\left[x^{\underbrace{\pi-0.5}_{p}}\right]' - \left[x^3\right]'$$

$$= 2^{\pi}\cdot(\pi-0.5)\,x^{\pi-0.5-1} - 3\cdot x^{3-1}$$

$$= 2^{\pi}(\pi-0.5)\,x^{\pi-1.5} - 3x^2$$

Sine and Cosine

Theorem ① $\frac{d}{dx}(\sin x) = \cos x$ Don't forget

② $\frac{d}{dx}(\cos x) = -\sin x$

Why? ① Textbook

Recall $\cos(a+b) = \cos a \cos b - \sin a \sin b$

② $$\frac{d}{dx}(\cos x) = \lim_{h\to 0}\frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h\to 0}\frac{\cos x\cosh - \sin x \sinh - \cos x}{h}$$

$$= \lim_{h\to 0}\left[\cos x\underbrace{\left(\frac{\cosh - 1}{h}\right)}_{\to 0} - \sin x\underbrace{\left(\frac{\sinh}{h}\right)}_{\to 1}\right]$$

$$= -\sin x$$

Ex Find y' if $y = 2\sin x - 5\cos x$

$$y' = 2[\sin x]' - 5[\cos x]'$$

$$= 2\cos x - 5(-\sin x)$$

$$= 2\cos x + 5\sin x$$

Ex Find the equation of the tangent line for $f(x) = 3\sin x$ at $x = \pi$.

Slope: Need $f'(\pi)$

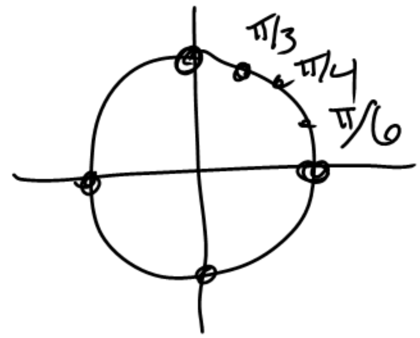
$$f'(x) = 3[\sin x]' = 3\cos x$$

$$f'(\pi) = 3\cos\pi = 3(-1) = -3$$

Point: $(\pi, f(\pi)) = (\pi, 3\sin\pi) = (\pi, 0)$

$$y - 0 = -3(x - \pi)$$

$$y = -3x + 3\pi$$



e^x:

$$\frac{d}{dx}(e^x) = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} \frac{e^x e^h - e^x}{h}$$

$$= e^x \underbrace{\lim_{h\to 0} \frac{e^h - 1}{h}}_{\to 1} = e^x$$

Theorem $y = e^x$ is the only that satisfies the following Initial Value Problem (IVP)

$$\begin{cases} \frac{dy}{dx} = y & \leftarrow \frac{d}{dx}(e^x) = e^x \\ y(0) = 1 & \leftarrow e^0 = 1 \end{cases}$$

Ex What is $y'(0)$ if $y = 5e^x + 7$

$$y' = 5[e^x]' + [7]'$$

$y' = 5e^x + 0 = 5e^x$ $\qquad e^{\boxed{2x}}$

$y'(0) = 5 \cdot e^0 = \boxed{5}$

Ex/ When does $\frac{dy}{dx} = 1$ if $y = 72e^x$

$\frac{dy}{dx} = 72 \cdot \frac{d}{dx}(e^x) = 72e^x \overset{\text{set}}{=} 1$

$$72e^x = 1$$

$$e^x = \frac{1}{72}$$

$$\ln(e^x) = \ln\left(\frac{1}{72}\right)$$

$$x = \ln\left(\frac{1}{72}\right)$$

$$x = -\ln(72^{-1})$$

$$\boxed{x = -\ln 72}$$

Ex (HW 6, # 9) Find the equation of the tangent line $f(x) = \frac{x^4}{8} - \frac{16}{x^2}$ at $x = 2$

Slope: Need $f'(2)$

$$f(x) = \frac{1}{8}x^4 - 16x^{-2}$$

$$f'(x) = \frac{1}{8}[x^4]' - 16[x^{-2}]'$$

$$= \frac{1}{8} \cdot 4x^3 - 16 \cdot (-2x^{-3})$$

$$f'(x) = \frac{1}{2}x^3 + \frac{32}{x^3}$$

$$f'(2) = \frac{1}{2}\cdot 8 + \frac{32}{8}$$
$$= 4 + 4 = 8$$

y-coord: $f(2) = \frac{2^4}{8} - \frac{16}{2^2} = 2 - 4 = -2$

Equation: $y - f(2) = f'(2)\,(x-2)$

$$y - (-2) = 8(x-2)$$
$$y+2 = 8(x-2)$$
$$y+2 = 8x - 16$$
$$y = 8x - 18$$