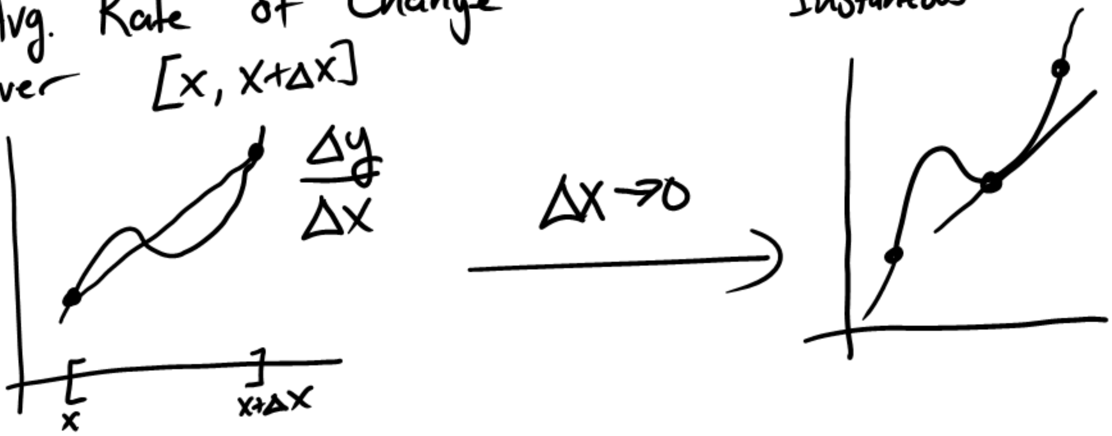
Lecture 7 - Instantenous Rates of Change

Goal: Interpret the role of the derivative in various contexts

Avg. Rate of Change over $[x, x+\Delta x]$ — Instaneous Rate of Change



Physics:

Instantenous Velocity: Avg. Velocity $= \frac{\Delta \text{Position}}{\Delta \text{Time}}$, so if $s(t)$ represents the position of an object at time $= t$, the instantenous velocity $v(t)$ is

$$v(t) = \frac{ds}{dt}$$

☆ Unless otherwise stated, t represents time in seconds

Ex/ The position of a particle moving in a straight line is given by

$$s(t) = 7\cos t + t^2$$

S is in meters

$$v(t) = \frac{ds}{dt} = \frac{d}{dt}\left(7\cos t + t^2\right) = 7\frac{d}{dt}(\cos t) + \frac{d}{dt}(t^2)$$

$$= 7(-\sin t) + 2t = (2t - 7\sin t)\,\frac{m}{s}$$

☆ Units of the derivative: $\frac{\text{Unit of the dependent var.}}{\text{Unit of the independent var.}}$

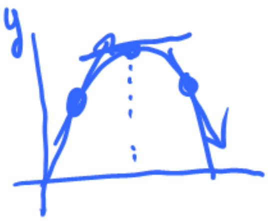
Ex/ A ball is tossed straight up in the air. It's height from the ground (in ft) can be modeled using the equation

$$S(t) = 80t - 16t^2$$

ⓐ Find its velocity

$$v(t) = \frac{ds}{dt} = \frac{d}{dt}(80t - 16t^2) = (80 - 32t)\,\frac{ft}{s}$$

ⓑ When $t = 3$, is the ball going up or down



$$v(3) = 80 - 32(3) = 80 - 96 = -16\ ft/s$$

Falling Downwards as $v(3) < 0$

ⓒ When does the ball reach its peak?

At the peak, the velocity is 0.

$$v(t) = 80 - 32t \overset{set}{=} 0$$

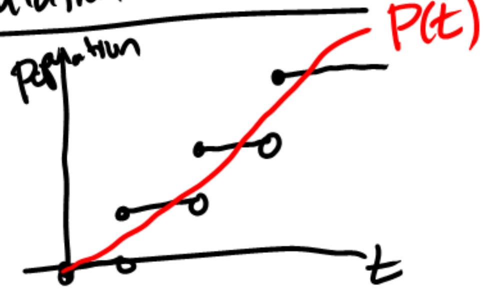
$$80 - 32t = 0$$

$$80 = 32t$$

$$t = \frac{80}{32} = 2.5 \text{ seconds}$$

Biology

Population Growth



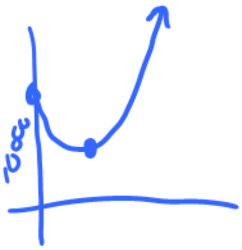
$P(t)$ is our population at a time t

Commonly: $P(t) = \underbrace{P_0}_{\text{Initial Population}} e^{\overset{\text{A constant}}{r}t}$

$\frac{dP}{dt}$ is called the <u>Growth Rate</u>; measure how fast the population is growing / shrinking.

Ex/ The population since 2000 can be modeled via the equation

$$P(t) = 100t^2 - 600t + 10000$$



ⓐ What is the rate at which the population is growing?

$$\frac{dP}{dt} = \frac{d}{dt}\left(100t^2 - 600t + 10000\right)$$

$$= (200t - 600)\ \frac{\#\text{ of people}}{\text{second}}$$

ⓑ When is the population decreasing?

Decreasing from the start to the vertex

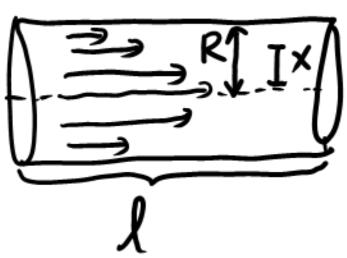
$$\frac{dP}{dt} = 200t - 600 \overset{\text{set}}{=} 0$$

$$200t = 600$$

$$t = 3$$

Decreasing when $t \in (0,3)$, i.e., from 2000 to 2003.

Blood Flow The velocity of the blood is given by



$$V(x) = \frac{P}{4\eta l}\left(R^2 - x^2\right)$$

← Pressure Difference

Viscocity

length

$\frac{dv}{dx}$ is called the velocity gradient

Ex/ In a small artery, the velocity can be modeled using

$$v(x) = (1.85 \times 10^4)(6.4 \times 10^{-5} - x^2)$$

x is in cm

What's the velocity gradient?

$$\frac{dv}{dx} = (1.85\times10^{4})\cdot\frac{d}{dx}\left(6.4\times10^{-5} - x^2\right)$$

$$= -2(1.85\times10^{4})x$$

(b) Evaluate at $x = 0.002$ cm

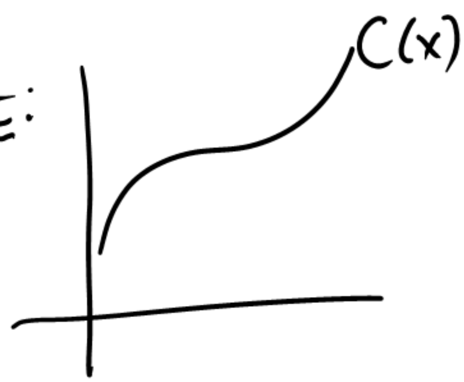
$$\left.\frac{dv}{dx}\right]_{0.002} = -2(1.85\times10^{4})(2\times10^{-3}) = -74\ \frac{(\text{cm/s})}{\text{cm}}$$

Convert to μm ($1\text{cm} = 10000\,\mu m$)

$$\frac{-74\ \frac{\text{cm}}{\text{s}}}{1\text{cm}} = \frac{-74\cdot10000\ \frac{\mu m}{s}}{1\cdot10000\ \mu m} = -74\ \frac{\mu m/s}{\mu m}$$

Economics:

Marginal Cost:



The cost function $C(x)$ represents the cost of producing x items.

$C(0)$ is the overhead cost.

$\frac{dC}{dx}$ is called the marginal cost.

Ex/ In USD, a company has estimated that the cost of producing x items is

$$C(x) = 10000 + 5x + 0.01x^2$$

(a) What is the marginal cost at $x = 500$?

$$\frac{dC}{dx} = 0 + \frac{d}{dx}(5x) + \frac{d}{dx}(0.01x^2) = 5 + 0.02x$$

$$\left.\frac{dC}{dx}\right]_{x=500} = 5 + 0.02(500) = 5 + 10 = \$15/\text{item}$$

Note: $C'(500) \approx C(501) - C(500) = \$15.01$

<u>Marginal Profit</u>: Profit = Revenue − Cost

$$P(x) = R(x) - C(x)$$

EX/ If the revenue is $R(x)=x$ and the cost is $C(x) = 50 - 0.01x^2$, what is the marginal profit?

$$P'(x) = R'(x) - C'(x)$$
$$= (x)' - (50 - 0.01x^2)'$$
$$= 1 - (-0.02x) = 1 + 0.02x$$