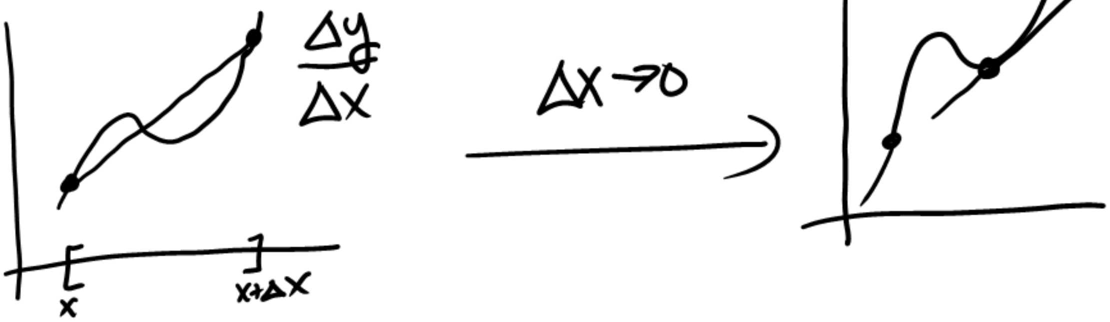
Lecture 7 - Instantenous Rates of Change

Goal: Interpret the role of the derivative in varius contexts

Avg. Rate of Change over $[x, x+\Delta x]$ — Instaneous Rate of Change



Physics

Instantenous Velocity:

$$\text{velocity} = \frac{\text{Change in Position}}{t} = \frac{\Delta s(t)}{\Delta t}$$

If $s(t)$ represents the position of an object at time $=t$, the velocity $v(t)$ is

$$v(t) = \frac{ds}{dt}$$

☆ Unless otherwise stated, t represents time in seconds

Ex/ The position of a particle (in meters) moving in a straight line can be modeled via

$$s(t) = 7\cos t + t^2$$

What's the velocity?

$$v(t) = \frac{ds}{dt} = \frac{d}{dt}\left(7\cos t + t^2\right) = 7\frac{d}{dt}(\cos t) + \frac{d}{dt}(t^2)$$

$$= 7(-\sin t) + 2t = (2t - 7\sin t)\,\frac{m}{s}$$

☆ Unit of Derivative: $\dfrac{\text{Unit of Dependent Variable}}{\text{Unit of Independent Variable}}$

Ex/ A ball is tossed straight into the air. The height from the ground (in feet) can be modeled via

$$s(t) = 80t \underbrace{-16}_{-\frac{1}{2}g} t^2$$

(a) What's the velocity?

$$v(t) = \frac{ds}{dt} = \frac{d}{dt}(80t - 16t^2) = (80 - 32t)\ \frac{ft}{s}$$

(b) When $t=3$, is the ball going up or down?

$$v(3) = 80 - 32(3) = 80 - 96 = -16\ \frac{ft}{s}$$

$v(3) < 0$, so the ball is going down.

(c) When does the ball reach its peak?

$$v(t) = 80 - 32t \overset{\text{set}}{=} 0$$

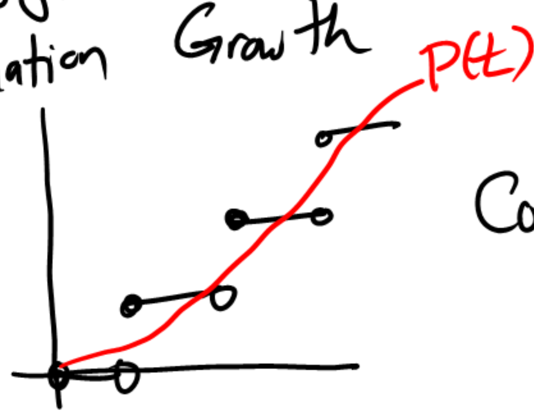
$$80 - 32t = 0$$

$$80 = 32t$$

$$t = \frac{80}{32} = 2.5 \text{ seconds}$$

Biology

Population Growth



$P(t)$ represents the population at a time t.

Commonly: $P(t) = \underbrace{P_0}_{\text{Initial Population}} e^{\overset{\text{Some constant}}{r}t}$

$\frac{dP}{dt}$ is called the <u>population growth</u>; measures how fast a population is growing or shrinking.

Ex/ The population of a city since 2000 can be modeled via the equation

$$P(t) = 100t^2 - 600t + 10000$$

ⓐ What is the rate at which the population grows?

$$P'(t) = (100t^2 - 600t + 10000)'$$
$$= (200t - 600)\ \frac{\text{People}}{\text{year}}$$

ⓑ When is the city losing population?

$$P'(t) = 200t - 600 \overset{\text{set}}{=} 0$$
$$200t - 600 = 0$$
$$200t = 600$$
$$t = 3 \text{ years}$$

Decreasing when $t \in (0,3)$, ie, from 2000 to 2003.

Blood Flow

The velocity of blood is modeled via the equation

$$V(x) = \frac{P}{4\eta l}(R^2 - x^2)$$

Pressure Difference

Viscosity length

Radius

Distance from Center axis

$\frac{dv}{dx}$ is called the velocity gradient

Ex/ In a small artery, the velocity of the blood can be modeled via

$$v(x) = (1.85 \times 10^4)(6.4 \times 10^{-5} - x^2)$$

x is in cm

What is the velocity gradient?

$$v'(x) = (1.85\times10^4)\cdot\left(6.4\times10^{-5} - x^2\right)'$$

$$= -2(1.85\times10^4)x$$

(b) Evaluate at $x = 0.002\text{ cm} = 2\times10^{-3}\text{ cm}$

$$v'(0.002) = -2(1.85\times10^4)(2\times10^{-3}) = -74\,\frac{(\text{cm/s})}{\text{cm}}$$

$1\text{ cm} = 10000\,\mu\text{m}$

$$\frac{-74\text{ cm/s}}{1\text{ cm}} = \frac{-74\cdot10000\,\mu\text{m/s}}{1\cdot10000\,\mu\text{m}} = -74\,\frac{\mu\text{m/s}}{\mu\text{m}}$$

Economics

Marginal Cost

The cost function $C(x)$ represents the cost to produce x items.

$C(0)$ is the overhead costs.

$\frac{dC}{dx}$ is called the marginal cost.

EX/ In USD, a company estimates the cost of producing x items as

$$C(x) = 10000 + 5x + 0.01x^2$$

(b) Marginal cost at $x=500$?

$$C'(x) = 5 + 0.02x$$

$$C'(500) = 5 + 0.02(500) = \$15/\text{item}$$

NOTE: $C'(500) \approx C(501) - C(500) = 15.01$

Marginal Profit:

$$\text{Profit} = \text{Revenue} - \text{Cost}$$

$$P(x) = R(x) - C(x)$$

Ex/ If the revenue is $R(x) = x$ and the cost is $C(x) = 50 - 0.01x^2$, what is the marginal profit?

$$P'(x) = [R(x) - C(x)]' = R'(x) - C'(x)$$

$$= [x]' - [50 - 0.01x^2]'$$

$$= 1 - (-0.02x) = \$(1 + 0.02x)/\text{item}$$