

# Lecture 8 The Product Rule

Goal: Differentiate functions of the form  $f(x)g(x)$

NOTE:  $\frac{d}{dx}(f(x)g(x)) \neq \frac{d}{dx}(f(x)) \frac{d}{dx}(g(x))$

$$\frac{d}{dx}(x \cdot 1) = \frac{d}{dx}(x) = 1 \quad \text{X}$$

$$\frac{d}{dx}(x) \cdot \frac{d}{dx}(1) = 1 \cdot 0 = 0$$

$$\frac{d}{dx}(f(x)g(x)) = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x)g(x+\Delta x) - g(x+\Delta x)f(x) + g(x+\Delta x)f(x) - f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \left[ \underbrace{g(x+\Delta x)}_{\Delta x} \frac{f(x+\Delta x) - f(x)}{\Delta x} + \frac{g(x+\Delta x) - g(x)}{\Delta x} f(x) \right]$$

$$= g(x)f'(x) + f(x)g'(x)$$

Theorem Let  $f$  and  $g$  be differentiable. Then

$$\frac{d}{dx}(f(x)g(x)) = \frac{d}{dx}(f(x))g(x) + f(x)\frac{d}{dx}(g(x))$$

$$(f(x)g(x))' = \underbrace{f'(x)}_{\text{"left"}} \underbrace{g(x)}_{\text{"right"}} + \underbrace{f(x)}_{\text{right d-left}} \underbrace{g'(x)}_{\text{left d-right}}$$

*right d-left + left d-right*

This is called the product rule.

Ex/ Compute  $\frac{d}{dx}(x \sin x) = \frac{d}{dx}(x) [\sin x] + x \cdot \frac{d}{dx}(\sin x)$

$$= 1 \cdot \sin x + x \cdot \cos x$$
$$= \sin x + x \cos x$$

Ex/ Compute  $(\underbrace{2 \sin x}_f \underbrace{\cos x}_g)' = [2 \sin x]' \cos x + 2 \sin x [\cos x]'$

$$= (2 \cos x)(\cos x) + 2 \sin x (-\sin x)$$

$$= 2(\cos^2 x - \sin^2 x)$$

Ex/ Compute  $(\underbrace{2}_f \underbrace{\sin x \cos x}_g)' = [2]' \sin x \cos x + 2 [\sin x \cos x]'$

$$= 2 [\underbrace{\sin x}_f \underbrace{\cos x}_g]'$$

$(2 \sin x \cos x) \quad [(2x)^5 \cos x]$

$$= 2 [\cos^2 x + \sin x (-\sin x)]$$

Ex/ Compute  $h'$  if  $h(x) = x^4 (3x^3 + 4x + 1)$

$$f(x) = x^4 \rightarrow f'(x) = 4x^3$$

$$g(x) = 3x^3 + 4x + 1 \rightarrow g'(x) = 9x^2 + 4$$

$$h'(x) = [f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$$

$$= (4x^3)(3x^3 + 4x + 1) + x^4(9x^2 + 4)$$

Ex/ Compute  $h'$  if  $h(x) = \underbrace{(x^{10} + x + 1)}_f \underbrace{(x^2 + 9)}_g$

$$f(x) = x^{10} + x + 1 \rightarrow f'(x) = 10x^9 + 1$$

$$g(x) = x^2 + 9 \rightarrow g'(x) = 2x$$

$$h'(x) = f'(x)g(x) + f(x)g'(x)$$

$$= (10x^9 + 1)(x^2 + 9) + (x^{10} + x + 1)(2x)$$

Ex/ Compute  $h'$  if  $h(x) = 6e^x \sin x - 13e^x \cos x$   
 $= \underbrace{e^x}_f \underbrace{(6 \sin x - 13 \cos x)}_g$

$$\begin{aligned} h'(x) &= [e^x]' (6 \sin x - 13 \cos x) + e^x [6 \sin x - 13 \cos x]' \\ &= e^x (6 \sin x - 13 \cos x) + e^x [6 \cos x - 13(-\sin x)] \\ &= e^x [6 \sin x - 13 \cos x + 6 \cos x + 13 \sin x] \\ &= e^x [19 \sin x - 7 \cos x] \end{aligned}$$

Ex/ Compute  $h'$  if  $h(x) = \underbrace{(x+1)}_f \underbrace{(2x+1)(3x+1)}_g$  ~~\*~~

$$\begin{aligned} h'(x) &= [x+1]' (2x+1)(3x+1) + (x+1) \underbrace{[(2x+1)(3x+1)]'}_{f \cdot g} \quad \text{~~*~~} \\ &= [x+1]' (2x+1)(3x+1) + (x+1) \left[ [2x+1]' (3x+1) + (2x+1)[3x+1]' \right] \\ &= 1 \cdot (2x+1)(3x+1) + (x+1) [2(3x+1) + 3(2x+1)] \\ &= (2x+1)(3x+1) + 2(x+1)(3x+1) + 3(x+1)(2x+1) \end{aligned}$$

Ex/ Find the equation of the tangent line if

$$h(x) = \underbrace{(\sqrt{x} + 4)}_f \underbrace{(x^{\frac{2}{3}} - 2x)}_g \quad \text{at } x=1$$

Slope  $[h'(1)]$

$$h'(x) = [\sqrt{x} + 4]' (x^{\frac{2}{3}} - 2x) + (\sqrt{x} + 4) [x^{\frac{2}{3}} - 2x]'$$

$$\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}(x^{\frac{2}{3}}) = \frac{2}{3}x^{\frac{2}{3}-1} = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}$$

$$h'(x) = \left(\frac{1}{2\sqrt{x}}\right)(x^{\frac{2}{3}} - 2x) + (\sqrt{x} + 4)\left(\frac{2}{3\sqrt[3]{x}} - 2\right)$$

$$h'(1) = \left(\frac{1}{2}\right)(1-2) + (1+4)\left(\frac{2}{3} - 2\right) = -\frac{43}{6}$$

Fractions,  
am I right?

Point:

$$(1, h(1)) \quad h(x) = (\sqrt{x} + 4)(x^{\frac{2}{3}} - 2)$$

$$h(1) = (1+4)(1-2) = -5$$

$$(1, h(1)) = (1, -5)$$

$$\text{Equation: } y - (-5) = -\frac{43}{6}(x-1)$$

Ex/ When does  $y = 10x^5 e^x$  have a horizontal tangent line?

$$y' = 50x^4 e^x + 10x^5 e^x = 10x^4 e^x (5+x) \stackrel{\text{set}}{=} 0$$

Either  $10x^4 = 0$   $\boxed{x=0}$  OR  $e^x = 0$  Impossible OR  $5+x=0$   $\boxed{x=-5}$