Goal: Differentiate functions of the form $\frac{f(x)}{g(x)}$. Use this to derive the derivatives of the 6 trigonometric functions.

Summary:

$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$		
$(\sin x)' = \cos x$	$(\cos x)' = -\sin x$	$(\tan x)' = \sec^2 x$
$(\sec x)' = \sec x \tan x$	$(\csc x)' = -\csc x \cot x$	$(\cot x)' = -\csc^2 x$

NOTE: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) \neq \frac{d}{dx}(f) \Big/ \frac{d}{dx}(g)$; $\frac{d}{dx}\left(\frac{x}{1}\right) \neq \frac{\frac{d}{dx}(x)}{\frac{d}{dx}(1)}$

$\frac{f(x)}{g(x)} = f(x)\left[\frac{1}{g(x)}\right]$. Need to know $\frac{d}{dx}\left(\frac{1}{g(x)}\right) = -\frac{g'(x)}{[g(x)]^2}$

Now by the product rule,

$$\left(\frac{f(x)}{g(x)}\right)' = \left[f(x)\cdot\frac{1}{g(x)}\right]' = f'(x)\cdot\frac{1}{g(x)} + f(x)\left(\frac{-g'(x)}{[g(x)]^2}\right)$$

$$= \frac{f'(x)}{g(x)}\left(\frac{g(x)}{g(x)}\right) - \frac{f(x)g'(x)}{[g(x)]^2}$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Theorem (Quotient Rule) Let f and g be differentiable and $g(x) \neq 0$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

"high" "Low"

"low d-high high d-low square the low"

Ex/ Find $\left(\frac{\overset{f}{1}}{\underset{g}{x}}\right)' = \frac{\cancel{[1]'\cdot x}^{\,0} - [1]\cdot[x]'}{x^2} = \frac{-1}{x^2}$

Ex/ Find $\left(\dfrac{\sin x}{x^2+x}\right)'$.

$f(x) = \sin x \to f'(x) = \cos x$

$g(x) = x^2+x \to g'(x) = 2x+1$

$$\left(\frac{\sin x}{x^2+x}\right)' = \frac{[\sin x]'(x^2+x) - \sin x\,[x^2+x]'}{(x^2+x)^2}$$

Ex/ $$\left(\frac{\overset{f}{2e^t}}{\underset{g}{e^t-5}}\right)' = \frac{[2e^t]'(e^t-5) - (2e^t)[e^t-5]'}{[e^t-5]^2}$$

$$= \frac{2e^t(e^t-5) - 2e^t(e^t)}{[e^t-5]^2}$$

$$= \frac{2e^{2t} - 10e^t - 2e^{2t}}{[e^t-5]^2} = -\frac{10e^t}{[e^t-5]^2}$$

Ex/ Let a be a constant

$$\left(\frac{x^2-a^2}{x-a}\right)' = \frac{[x^2-a^2]'(x-a) - (x^2-a^2)[x-a]'}{(x-a)^2}$$

$f \to x^2-a^2$, $g \to x-a$

$$= \frac{2x(x-a) - (x^2-a^2)}{(x-a)^2} = \frac{2x^2-2ax-x^2+a^2}{(x-a)^2}$$

NOTE: $\left(\dfrac{x^2-a^2}{x-a}\right)' = \left(\dfrac{(x+a)(x-a)}{x-a}\right)' = (x+a)' = 1$

$$= \frac{x^2-2ax+a^2}{(x-a)^2} = \frac{(x-a)^2}{(x-a)^2} = 1$$

Derivatives of Trig Functions

As a reminder, $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$

Tangent

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{[\sin x]'\cos x - \sin x\,[\cos x]'}{[\cos x]^2}$$

$$= \frac{\cos^2 x + \sin^2 x}{[\cos x]^2} = \frac{1}{[\cos x]^2} = \boxed{\sec^2 x}$$

Ex/ If $y = \underbrace{\cos x \tan x}_{=\sin x}$, what is y'? $\sin^2 x + \cos^2 x = 1$

$$y' = [\cos x]' \tan x + \cos x \boxed{[\tan x]'}$$

$$= -\sin x \tan x + \cos x \boxed{\sec^2 x}$$ ← You can stop here

$$= \frac{-\sin^2 x}{\cos x} + \frac{\cos x}{\cos^2 x} = \frac{-\sin^2 x}{\cos x} + \frac{1}{\cos x} = \frac{1 - \sin^2 x}{\cos x}$$

$$= \frac{\cos^2 x}{\cos x} = \cos x$$

Cotangent

$$\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{[\cos x]'\sin x - \cos x\,[\sin x]'}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{\sin^2 x + \cos^2 x}{\sin^2 x}$$

$$= -\frac{1}{\sin^2 x} = \boxed{-\csc^2 x}$$

Ex/ If $y = e^x \cot x$, then

$$y' = [e^x]' \cot x + e^x [\cot x]'$$

$$= e^x \cot x + e^x(-\csc^2 x) = e^x(\cot x - \csc^2 x)$$

Secant

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{\cancel{[1]'\cos x}^{\,0} - 1\cdot[\cos x]'}{[\cos x]^2} = \frac{\sin x}{[\cos x]^2}$$

$$= \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} = \boxed{\sec x \tan x}$$

Ex/ Find y' if $y = \sec^2 x = (\sec x)(\sec x)$

$$y' = [\sec x]'\sec x + \sec x\,[\sec x]' = (\sec x \tan x)\sec x + \sec x(\sec x \tan x)$$

$$= \underbrace{2\sec^2 x \tan x}_{\text{Simplified Form}} = (2\sec x)(\sec x \tan x)$$

Cosecant

$$\frac{d}{dx}(\csc x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = \frac{\cancel{[1]'\sin x}^{\,0} - 1\cdot[\sin x]'}{[\sin x]^2} = \frac{-\cos x}{[\sin x]^2}$$

$$= -\left(\frac{1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right) = \boxed{-\csc x \cot x}$$

Ex/ If $y = x^5 \csc x$, then

$$y' = [x^5]'\csc x + x^5[\csc x]'$$

$$= 5x^4\csc x + x^5(-\csc x \cot x)$$

$$= x^4 \csc x(5 - x\cot x)$$

Avoiding Tedious Work

Ex/ Find $g'(0)$ if $g(x) = \dfrac{\overbrace{5x^8 + 6x^5 + 5x^4 + 3x^2 + 20x + 100}^{f}}{\underbrace{10x^{10} + 8x^9 + 6x^5 + 6x^2 + 4x + 2}_{g}}$

$$q'(0) = \frac{f'(0)\,g(0) - f(0)\,g'(0)}{[g(0)]^2}$$

When differentiating a polynomial, the linear term becomes the constant term

$f(0) = 100$ $\quad f'(0) = 20$

$g(0) = 2$ $\quad g'(0) = 4$

$$q'(0) = \frac{f'(0)\,g(0) - f(0)\,g'(0)}{[g(0)]^2} = \frac{20(2) - 100(4)}{4}$$

$$= \frac{40 - 400}{4} = \frac{-360}{4} = -90$$