Goal: Differentiate functions of the form $\frac{f(x)}{g(x)}$. Use this to derive the derivative^s of the 6 trigonometric functions.

Summary:

$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$		
$(\sin x)' = \cos x$	$(\cos x)' = -\sin x$	$(\tan x)' = \sec^2 x$
$(\sec x)' = \sec x \tan x$	$(\csc x)' = -\csc x \cot x$	$(\cot x)' = -\csc^2 x$

NOTE: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) \neq \frac{d}{dx}(f) / \frac{d}{dx}(g)$; $\frac{d}{dx}\left(\frac{x}{1}\right) \neq \frac{d/dx(x)}{d/dx(1)}$

$$\frac{d}{dx}\left(f(x)/g(x)\right) = \frac{d}{dx}\left[f(x) \cdot \frac{1}{g(x)}\right]; \quad \frac{d}{dx}\left(\frac{1}{g(x)}\right) = -\frac{g'(x)}{[g(x)]^2}$$

By the Product Rule,

$$\left(\frac{f(x)}{g(x)}\right)' = \left[f(x) \cdot \frac{1}{g(x)}\right]' = f'(x) \cdot \frac{1}{g(x)} + f(x)\left[\frac{1}{g(x)}\right]'$$

$$= \frac{f'(x)}{g(x)} + f(x)\left[\frac{-g'(x)}{[g(x)]^2}\right]$$

$$= \frac{f'(x)}{g(x)}\left(\frac{g(x)}{g(x)}\right) - \frac{f(x)\,g'(x)}{[g(x)]^2}$$

$$= \frac{f'(x)\,g(x) - f(x)\,g'(x)}{[g(x)]^2}$$

Theorem (Quotient Rule) Let f and g be differentiable and $g(x) \neq 0$

"high" → $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ "low d-high high d-low square the low"

"low" →

Ex/ Find $\left(\frac{1}{x}\right)' = \frac{[1]'^{\,0}\,x - 1 \cdot [x]'}{x^2} = \frac{-1}{x^2}$

f = 1, g = x

Ex/ Find $\left(\dfrac{\sin x}{x^2+x}\right)'$. $f(x)=\sin x \longrightarrow f'(x)=\cos x$

$g(x)=x^2+x \longrightarrow g'(x)=2x+1$

$$\left(\frac{\sin x}{x^2+x}\right)' = \frac{[\sin x]'(x^2+x) - \sin x[x^2+x]'}{[x^2+x]^2}$$

$$= \frac{\boxed{\cos x}(x^2+x) - \boxed{\sin x}(2x+1)}{[x^2+x]^2}$$

Ex/ Find $\left(\dfrac{\overset{f}{\boxed{2e^t}}}{\underset{g}{\boxed{e^t-5}}}\right)' = \dfrac{[2e^t]'(e^t-5) - (2e^t)[e^t-5]'}{(e^t-5)^2}$

$$= \frac{2e^t(e^t-5) - 2e^t(e^t)}{(e^t-5)^2}$$

$$= \frac{2e^{2t} - 10e^t - 2e^{2t}}{(e^t-5)^2} = -\frac{10e^t}{(e^t-5)^2}$$

Ex/ Let a be a constant

$$\left(\frac{\overbrace{x^2-a^2}^{f}}{\underbrace{x-a}_{g}}\right)' = \frac{[x^2-a^2]'(x-a) - (x^2-a^2)[x-a]'}{(x-a)^2}$$

$$= \frac{2x(x-a) - (x^2-a^2)}{(x-a)^2}$$

$$= \frac{2x^2 - 2ax - x^2 + a^2}{(x-a)^2}$$

$$= \frac{x^2 - 2ax + a^2}{(x-a)^2} = \frac{(x-a)^2}{(x-a)^2} = 1$$

NOTE: $\dfrac{x^2-a^2}{x-a} = \dfrac{(x+a)(x-a)}{(x-a)} = x+a$

Derivatives Of Trig Functions

Recall $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$

Tangent

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\overbrace{\sin x}^{f}}{\underbrace{\cos x}_{g}}\right) = \frac{[\sin x]'\cos x - \sin x\,[\cos x]'}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \boxed{\sec^2 x}$$

Ex/ If $y = \cos x \tan x$, $\left(\tan x = \frac{\sin x}{\cos x}\right)$ What is y'?

You can stop here

$$y' = [\cos x]'\tan x + \cos x\,[\tan x]' = -\sin x \tan x + \cos x \sec^2 x$$

$$= \frac{-\sin^2 x}{\cos x} + \frac{\cos x}{\cos^2 x} = \frac{-\sin^2 x}{\cos x} + \frac{1}{\cos x} = \frac{1 - \sin^2 x}{\cos x}$$

$$= \frac{\cos^2 x}{\cos x} = \cos x$$

$$\cancel{\sin^2 x} + \cos^2 x = 1$$
$$\quad -\cancel{\sin^2 x} \qquad -\sin^2 x$$
$$\cos^2 x = 1 - \sin^2 x$$

Cotangent

$$\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{[\cos x]'\sin x - \cos x\,[\sin x]'}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x}$$

$$= \boxed{-\csc^2 x}$$

Ex/ If $y = e^x \cot x$, then

$$y' = [e^x]'\cot x + e^x[\cot x]' = e^x \cot x + e^x(-\csc^2 x)$$

$= e^x(\cot x - \csc^2 x)$

Secant

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{\overset{0}{\cancel{[1]'\cos x}} - 1\cdot[\cos x]'}{\cos^2 x} = \frac{\sin x}{\cos^2 x}$$

$$= \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right) = \boxed{\sec x \tan x}$$

Ex/ If $y = \sec^2 x = (\sec x)(\sec x)$, then

$$y' = [\sec x]'\sec x + \sec x[\sec x]'$$

$$= (\sec x \tan x)\sec x + \sec x(\sec x \tan x) = \overbrace{2\sec^2 x \tan x}^{\text{Simplified}}$$

$$= (2\sec x)(\sec x \tan x)$$

Cosecant

$$\frac{d}{dx}(\csc x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = \frac{\overset{0}{\cancel{[1]'\sin x}} - 1\cdot[\sin x]'}{\sin^2 x} = \frac{-\cos x}{\sin^2 x}$$

$$= -\left(\frac{1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right) = \boxed{-\csc x \cot x}$$

Ex/ Find $(x^5 \csc x)'$

$$(\underbrace{x^5}_{f}\underbrace{\csc x}_{g})' = [x^5]'\csc x + x^5[\csc x]'$$

$$= 5x^4\csc x + x^5(-1)\csc x \cot x$$

$$= x^4 \csc x(5 - x\cot x)$$

Avoiding Tedious Work

Ex/ Find $g'(0)$ if

$$q(x) = \frac{\overbrace{5x^8 + 6x^5 + 5x^4 + 3x^2 + 20x + 100}^{f}}{\underbrace{10x^{10} + 8x^9 + 6x^5 + 6x^2 + 4x + 2}_{g}}$$

$$q'(x) = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{[g(x)]^2}$$

$$q'(0) = \frac{f'(0)\,g(0) - f(0)\,g'(0)}{[g(0)]^2}$$

$f(0) = 100$

$g(0) = 2$

$f'(0) =$ The constant term of f'

$=$ The linear term of f

$= 20$

$g'(0) = 4$

$$= \frac{20(2) - 100(4)}{4}$$

$$= \frac{40 - 400}{4} = 10 - 100 = -90$$