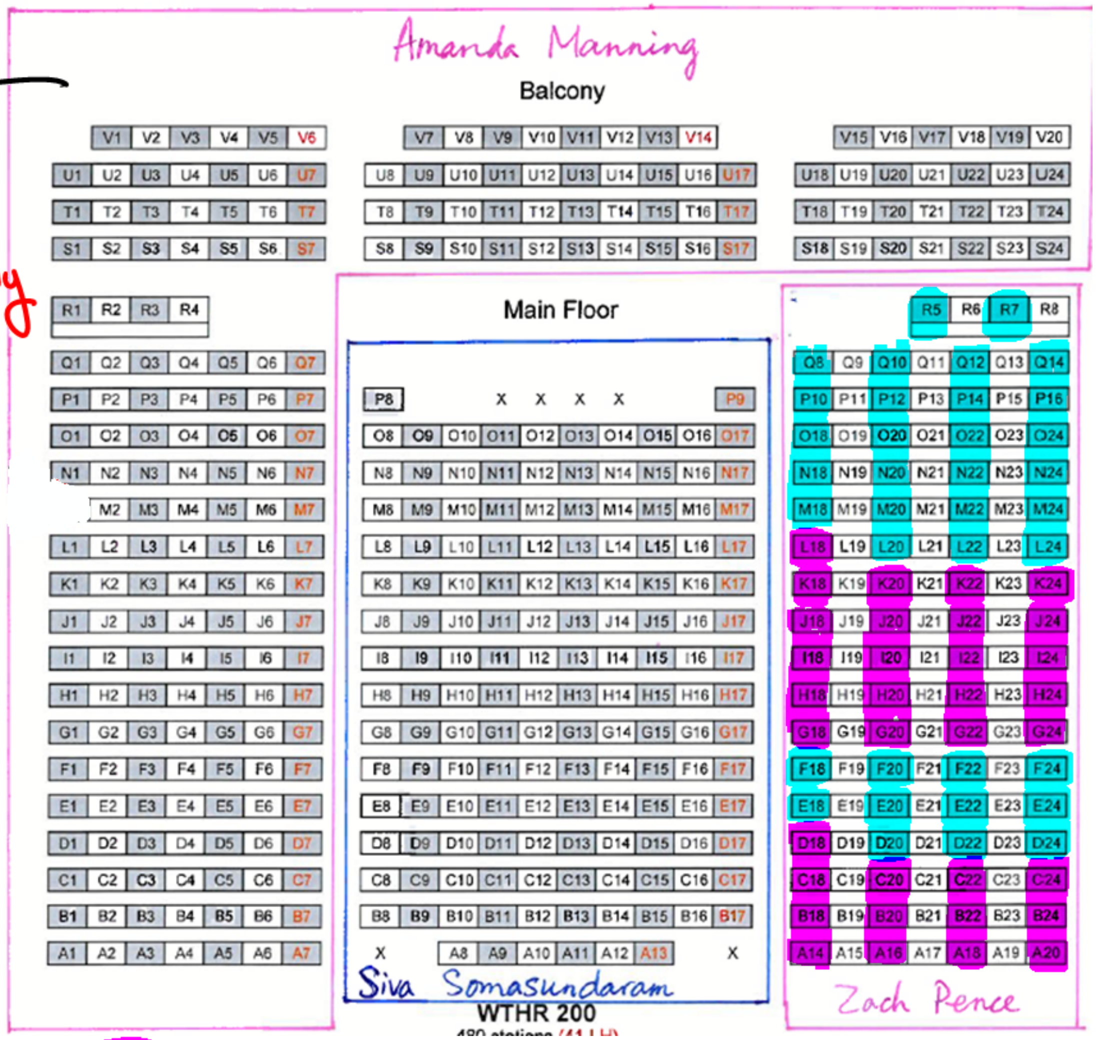
MT 3 Review

- Monday (11/10)
- 8-9PM
- WTHR 200 ← Main Floor NOT Balcony
  - Lectures 20-28

Make sure to bring
- Pencils / Erasers
- PUID
- Calculator
  TI 30XA
  Single line only



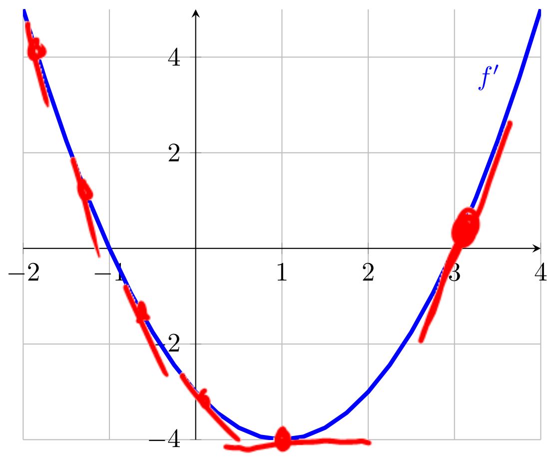
7:30 Section 8:30 Section

Problem 1. *A graph of f' is given below.*

(a) *Determine when f is increasing and when it is decreasing.*

(b) *Determine when f is concave up and when it is concave down.*

(c) *Locate the positions (x-coordinates) of any relative extrema and inflection points.*



Inc/Dec:

$f'>0$ -1 $f'<0$ 3 $f'>0$

Inc: $(-\infty,-1) \cup (3,\infty)$

Dec: $(-1,3)$

Rel Max @ $x=-1$

Rel Min @ $x=3$

Concavity/IPs: $[f']' = f'' = 0$ @ $x=1$

$f''<0 \Rightarrow$ CD 1 $f''>0 \Rightarrow$ CU

CD on $(-\infty,1)$

CU on $(1,\infty)$

IP at $x=1$

Problem 2. *Compute* $\lim_{x\to\infty} \frac{x^2+1}{2-x}$ *and* $\lim_{x\to-\infty} \frac{x^2+1}{2-x}$.

$$\frac{x^2+1}{-x+2} = \frac{x\left(x+\frac{1}{x}\right)}{x\left(-1+\frac{2}{x}\right)} \overset{\text{when } x\neq 0}{=\!=} \frac{x+\frac{1}{x}}{-1+\frac{2}{x}}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\overset{\infty}{x} + \overset{\to 0}{\frac{1}{x}}}{\underset{-1}{-1} + \underset{\to 0}{\frac{2}{x}}} = -\infty$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x + 1/x}{-1 + 2/x} = \infty$$

Problem 3. *Use the techniques learned in Lecture 22 to sketch a graph of the function* $y = \frac{x^2}{x+8}$.

$\frac{\overbrace{x^2}^{\text{deg } 2}}{\underbrace{x+8}_{\text{deg } 1}}$

Domain: $\mathbb{R} \setminus \{-8\}$

Intercepts: $(0,0)$. $\quad \frac{x^2}{x+8} \overset{\text{set}}{=} 0 \Rightarrow x^2 = 0 \Rightarrow x = 0$

Asymptotes: VA @ $x = -8$

Slant: $y = x - 8$

$$\begin{array}{r} x - 8 \\ x+8 \overline{) \; x^2 \qquad} \\ -(x^2 + 8x) \\ \hline -8x \\ -(-8x - 64) \\ \hline 64 \end{array}$$

Rel Max/Min:

$f(x) = \frac{x^2}{x+8}$

$f'(x) = \frac{2x(x+8) - x^2}{(x+8)^2} = \boxed{\frac{x^2 + 16x}{(x+8)^2}} \overset{\text{set}}{=} 0$

$\Rightarrow x^2 + 16x = 0 \Rightarrow x(x+16) = 0 \Rightarrow x = -16, 0$

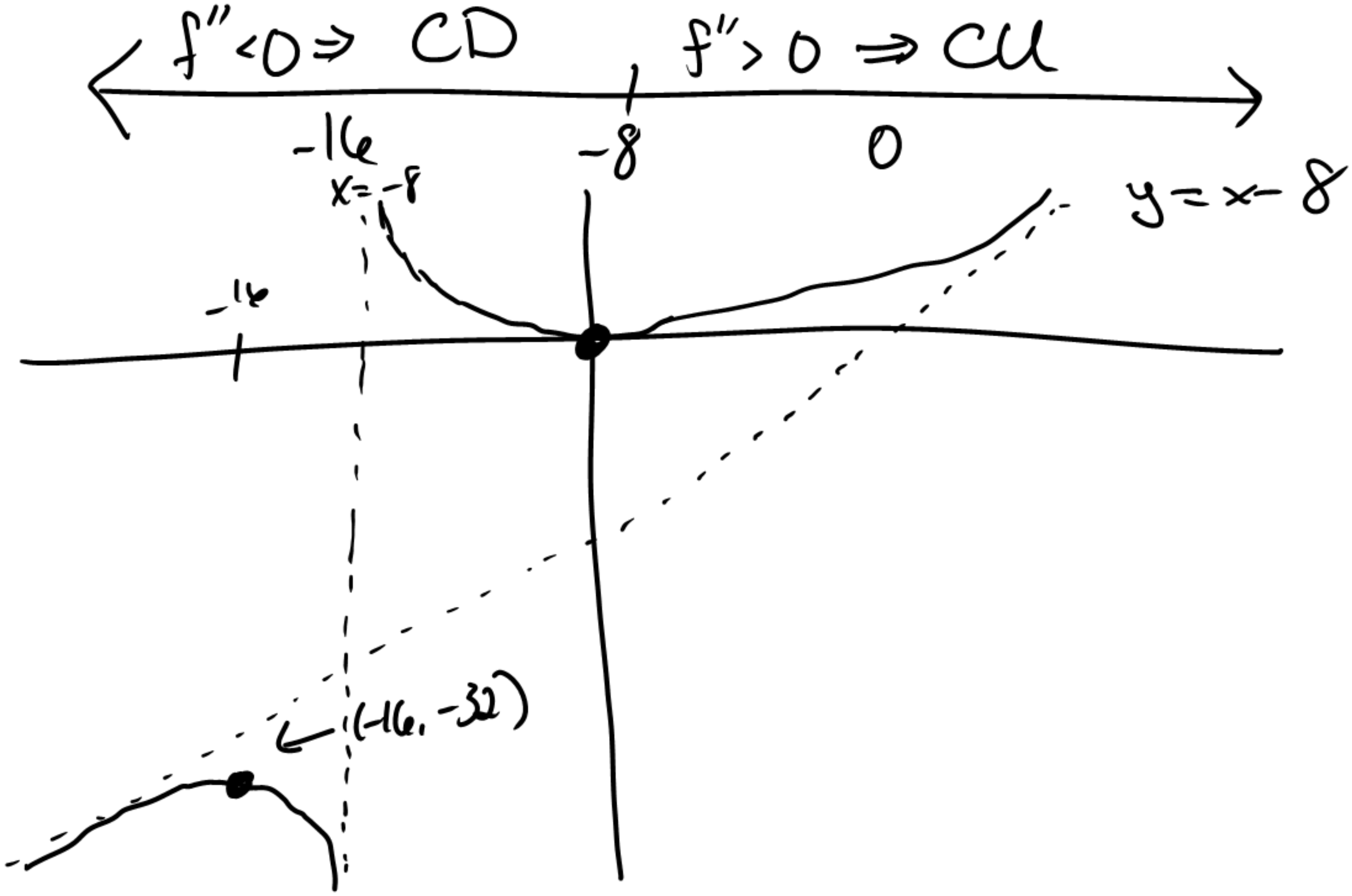
$$f''(x) = \frac{(2x+16)(x+8)^2 - (x^2+16x)2(x+8)}{(x+8)^4} = \frac{72(x+8)}{(x+8)^4}$$

$f''(-16) < 0 \Rightarrow$ Local Max @ $x = -16$

$f''(0) > 0 \Rightarrow$ Local Min @ $x = 0$

Concavity / IPs: $\frac{72(x+8)}{(x+8)^4} \overset{\text{set}}{=} 0 \Rightarrow 72(x+8) = 0 \Rightarrow x = -8$

Not IP, but concavity could change at an asymptote


f'' < 0 ⇒ CD
f'' > 0 ⇒ CU
-16
-8
0
x = -8
y = x - 8
-16
(-16, -32)

Problem 4. *If a rectangle has a fixed perimeter of 40, what is its maximum area?*

Obj: Maximize $A(x,y) = xy$

$$40 = 2x + 2y$$
$$20 = x + y$$
$$y = 20 - x$$

$$A(x) = x(20-x) = 20x - x^2$$

$$A'(x) = 20 - 2x \overset{\text{set}}{=} 0$$
$$x = 10$$

Verify it is an abs. max:

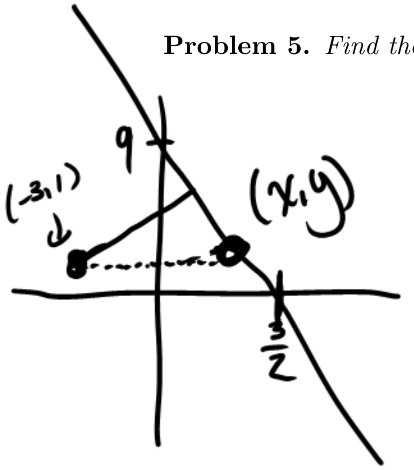
$$A''(x) = -2 < 0 \Rightarrow \begin{matrix}\text{Rel. Max at } x=10 \\ + \\ \text{Only 1 Crit. Num}\end{matrix} \Rightarrow \begin{matrix}\text{Abs. Max.} \\ \text{at } x=10\end{matrix}$$

$$A(10) = 10(20-10) = 10^2 = 100$$

Conclusion: The max area is 100

$$d=\sqrt{(x-x_0)^2+(y-y_0)^2}$$

Problem 5. *Find the point on the line $6x+y=9$ that is closest to the point $(-3,1)$.*



Obj: Min $s(x,y)=[d(x,y)]^2=(x+3)^2+(y-1)^2$

Given: $6x+y=9$

$y=-6x+9$

$$S(x)=(x+3)^2+(-6x+9-1)^2=(x+3)^2+(-6x+8)^2$$

$$S(x)=37x^2-90x+73$$

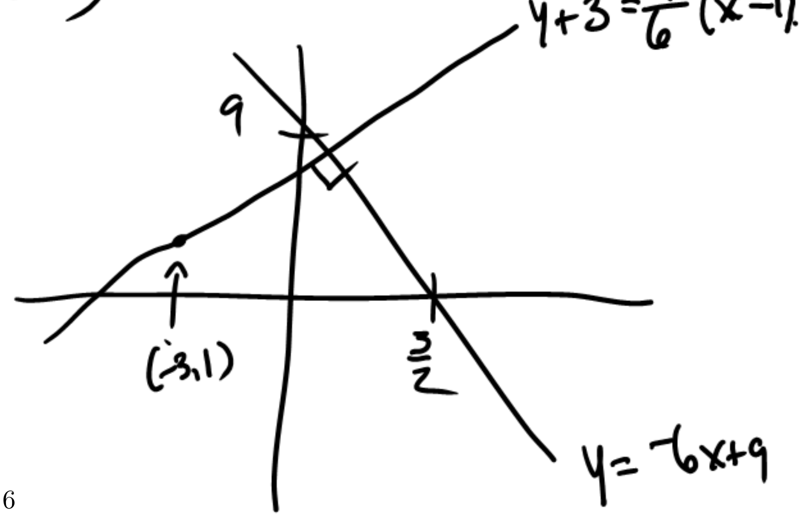
$$S'(x)=74x-90 \overset{\text{set}}{=} 0$$

$$x=\frac{90}{74}=\frac{45}{37}$$

Same Idea: $S''\left(\frac{45}{37}\right)=74>0$ + Only 1 Crit. Num $\Rightarrow$ Abs. Min.

$$y=-6\left(\frac{45}{37}\right)+9=\frac{63}{37}$$

Conclusion: The point $\left(\frac{45}{37},\frac{63}{37}\right)$ is closest to the point $(-3,1)$



Problem 6. *A particle is moving through space; its acceleration function is given by* $a(t) = \cos t + \sin t$ *for* $t \geq 0$. *Find the position of the particle at time* t *when* $s(0) = 0$ *and* $v(0) = 5$.

Need to solve the IVP

$$\begin{cases} s'' \to a(t) = \cos t + \sin t \\ s' \to v(0) = 5 \\ s(0) = 0 \end{cases}$$

$$a(t) = \cos t + \sin t$$

$$\int a(t)\,dt = \int (\cos t + \sin t)\,dt$$

$$v(t) = \sin t - \cos t + C$$

$v(0) = 5$:

$$5 = v(0) = 0 - 1 + C$$

$$5 = -1 + C$$

$$C = 6$$

$$v(t) = \sin t - \cos t + 6$$

$$\int v(t)\,dt = \int (\sin t - \cos t + 6)\,dt$$

$$s(t) = -\cos t - \sin t + 6t + D$$

$s(0) = 0$:

$$0 = s(0) = -1 - 0 + 0 + D$$

$$0 = -1 + D$$

$$D = 1$$

$$\boxed{s(t) = -\cos t - \sin t + 6t + 1}$$

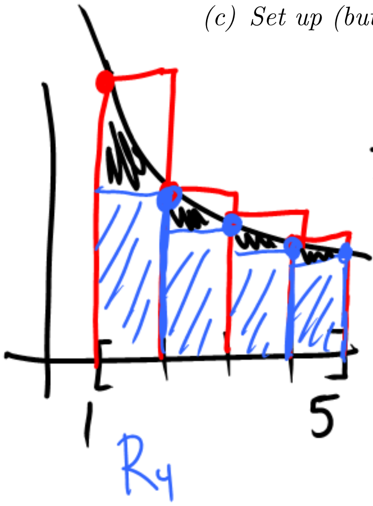
Problem 7. *We estimate the area underneath the graph of* $f(x) = \frac{1}{x}$ *on the interval* $[1, 5]$.

a, b

(a) *Compute the left Riemann sum with 4 rectangles. Is this an overestimate or an underestimate? Explain why (The exact area is* $\ln 5$*, but you don't need to know that to answer the question).*

(b) *Repeat (a) using the right Riemann sum.*

(c) *Set up (but do not compute) the left Riemann sum using N rectangles (where N is a positive integer).*



$\frac{1}{x}$

(a) Compute L_4

$$\Delta X = \frac{5-1}{4} = \frac{4}{4} = 1 \qquad \Delta X = \frac{b-a}{N}$$

$$L_4 = \sum_{i=0}^{3} f(1 + i(1))(1) = \sum_{i=0}^{3} \frac{1}{1+i}$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}$$ ← Over Estimate

(b) Compute R_4

$$R_4 = \sum_{i=1}^{4} \frac{1}{1+i} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{77}{60}$$ ← Under Estimate

Ie,

$$\frac{77}{60} < \ln(5) < \frac{25}{12}$$ ← You don't need to do this

(c) Find L_N

$$\Delta X = \frac{5-1}{N} = \frac{4}{N}$$

$$L_N = \sum_{i=0}^{N-1} f(a + i\,\Delta X)\,\Delta X \Longrightarrow L_N = \sum_{i=0}^{N-1} \frac{1}{1 + i\left(\frac{4}{N}\right)}\left(\frac{4}{N}\right)$$

↑ Left Endpoint