MA 161 ONLINE, SUMMER 2026
MIDTERM EXAM 2

Name (Print): KEY ____________________ PUID: ____________________

Section Number: ____________________ Name of TA: ____________________

Start Time: ____________________ End Time: ____________________

Directions:

- Please use a **pen** if possible. Marks made by pen scan better than marks made by pencil.
- This is a **one-hour** exam.
- There are **12 questions** worth 8 points each; the maximum score is 100 points.
- The remaining 4 points are earned for filling out the information at the top of the page correctly. Section numbers, and their associated TA, are provided in the table below.

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- Circle one and only one choice for each multiple-choice problem (Problems 1-3). **No partial credit** will be given on multiple-choice problems.
- Show all relevant work on each free-response problem (Problems 4-12). You must include some work or justification to receive full credit on these problems; partial credit may be given for steps leading to the correct solution. **Write the final answer in the box(es) provided**.
- Any external help or communication is not allowed, except with the proctor for logistical reasons. This exam is closed book and closed notes. **Calculators are not allowed**. Electronic devices, including smart watches, are to be turned off and put away.
- The last sheet is scratch paper. You may use only the scratch paper provided in this exam booklet. No additional scratch paper is permitted.

Purdue University faculty and students commit themselves towards maintaining a culture of academic integrity and honesty. The students taking this exam are not allowed to seek or obtain any kind of help from anyone to answer questions on this test. If you have questions, consult only an instructor or a proctor. You are not allowed to look at the exam of another student. You may not compare answers with anyone else or consult another student until after you finish your exam and hand it in to a proctor or to an instructor. You may not consult notes, books, calculators, cameras, or any kind of communications devices until after you finish your exam and hand it in to a proctor or to an instructor. If you violate these instructions you will have committed an act of academic dishonesty. Penalties for academic dishonesty can be very severe and may include an F in the course. All cases of academic dishonesty will be reported to the Office of the Dean of Students. Your instructor and proctors will do everything they can to stop and prevent academic dishonesty during this exam. If you see someone breaking these rules during the exam, please report it to the proctor or to your instructor immediately. Reports after the fact are not very helpful.

I agree to abide by the instructions above:

Student Signature: ____________________

Proctor Signature: ____________________

Question	Points	Score
1	8	
2	8	
3	8	
4	8	
5	8	
6	8	
7	8	
8	8	
9	8	
10	8	
11	8	
12	8	
Correct Header Info	4	
Total:	100	

1. [8 points] Let M and m be the maximum and minimum of $f(x) = 10 + 27x - x^3$ on the interval $[0, 4]$ respectively. What is the value of $M + m$?

   A. 20 [considered $x = -3$]

   B. 64 [Only checked endpoints]

   C. 74

   D. 118 [assumed $M = f(3), m = f(4)$]

   E. 128 [falsely performed $3^3 = 9$]

$f(x) = 10 + 27x - x^3$

$f'(x) = 27 - 3x^2 \overset{\text{set}}{=} 0$

$3(9 - x^2) = 0$

$3(3-x)(3+x) = 0$

$\Rightarrow x = \pm 3$

$\xrightarrow{x \in [0,4]} x = 3$

Check y value at 3 and endpoints

x	0	3	4
$f(x)$	10	64	54
	m	M	

$\Rightarrow M + m = 64 + 10 = 74$

2. [8 points] Evaluate the limit:

$$\lim_{x\to 0} \frac{\sin 3x}{\sin 7x}$$

A. 0 [assumes 0/0 means 0]

B. $\frac{3}{7}$

C. 1 [assumes 0/0 means 1]

D. $\frac{7}{3}$ [flipped coefficients]

E. The limit does not exist. [assumes 0/0 means DNE]

Recall $\lim_{x\to 0} \frac{\sin x}{x} = 1$ and in particular for $k \neq 0$

$$\lim_{x\to 0} \frac{\sin kx}{x} = k \lim_{x\to 0} \frac{\sin kx}{kx} \overset{u=kx}{=\!=\!=} k \lim_{u\to 0} \frac{\sin u}{u} = k$$

So,

$$\lim_{x\to 0} \frac{\sin 3x}{\sin 7x} = \lim_{x\to 0} \frac{\frac{\sin 3x}{x}}{\frac{\sin 7x}{x}} = \frac{3}{7}$$

3. [8 points] If $f(x) = \tan^{-1}(x^4 + 1)$, what is $f'(1)$?

A. f is not differentiable at $x = 1$ [uses d/dx of arcsin]

B. $-4\csc^2(2)$ [(roughly -4.83) assumed $\tan^{-1} x = \frac{1}{\tan x}$]

C. $\frac{1}{5}$ [Forgot chain rule]

D. $\frac{4}{5}$

E. $\frac{4}{3}$ [forgot square in d/dx of arctan]

$$f(x) = \tan^{-1}(x^4+1)$$

$$f'(x) = \frac{1}{1+(x^4+1)^2} \cdot \frac{d}{dx}(x^4+1) = \frac{4x^3}{1+(x^4+1)^2}$$

$$f'(1) = \frac{4 \cdot 1}{1+(1+1)^2} = \frac{4}{5}$$

4. Consider the function $f(x) = \frac{1}{x-1}$ on the interval $[0, 2]$.

(a) [6 points] Show there is **NOT** a point c between 0 and 2 such that $f'(c)$ is the average rate of change of f on $[0, 2]$.

Solution:

$$\begin{aligned} f'(c) &= \frac{f(2) - f(0)}{2 - 0} \\ -\frac{1}{(c-1)^2} &= \frac{1 - (-1)}{2 - 0} \\ (c-1)^2 &= -1 \end{aligned}$$

This equation has no solution.

(No answer box for Part (a).)

(b) [2 points] Why does this **NOT** contradict the Mean Value Theorem?

f is not differentiable at $x = 1$, hence it does not meet the conditions of the MVT.

5. [8 points] At what point is the line $y = 7 - 2x$ closest to the point $(1, 0)$?

For full points, use calculus to verify the point found is the location of a minimum. State the test used.

Minimize $S(x,y) = (x-1)^2 + y^2$

subject to $y = 7 - 2x$

$$S(x) = (x-1)^2 + (7-2x)^2$$

$$S'(x) = 2(x-1) + (-2)(2)(7-2x) \overset{\text{set}}{=} 0$$

$$2x - 2 - 4(7-2x) = 0$$

$$2x - 2 - 28 + 8x = 10$$

$$10x - 30 = 0$$

$$x = 3$$

Verify it is a minimum:

$S'(x) = 10x - 30$

$S''(x) = 10 < 0$ $\Rightarrow$ Location of a local minimum by the 2nd Derivative Test.

Since there is only one critical point on an open interval.

Local Min $\Rightarrow$ Global Min

y-coordinate: $y = 7 - 2(3) = 1$

So the point $(3,1)$ is closest to $(1,0)$

Point $(x, y) =$ $(3, 1)$

6. [8 points] Use Logarithmic Differentiation to find the derivative of

$$y = \frac{(x-2)^5\sqrt{x^2+1}}{(x+3)^7}$$

$$y = \frac{(x-2)^5\sqrt{x^2+1}}{(x+3)^7}$$

$$\ln y = \ln\left(\frac{(x-2)^5\sqrt{x^2+1}}{(x+3)^7}\right)$$

$$\ln y = 5\ln(x-2) + \frac{1}{2}\ln(x^2+1) - 7\ln(x+3)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{5}{x-2} + \frac{1}{2}\cdot\frac{2x}{x^2+1} - \frac{7}{x+3}$$

$$\frac{dy}{dx} = y\left[\frac{5}{x-2} + \frac{x}{x^2+1} - \frac{7}{x+3}\right]$$

$$\frac{dy}{dx} = \frac{(x-2)^5\sqrt{x^2+1}}{(x+3)^7}\left[\frac{5}{x-2} + \frac{x}{x^2+1} - \frac{7}{x+3}\right]$$

$$\boxed{\frac{dy}{dx} = \frac{(x-2)^5\sqrt{x^2+1}}{(x+3)^7}\left[\frac{5}{x-2} + \frac{x}{x^2+1} - \frac{7}{x+3}\right]}$$

7. [8 points] If $f(x) = 2^x + \log_3 x + \tan x + \csc x$, then what is $f'(x)$?

$f(x) = 2^x + \log_3 x + \tan x + \csc x$

$f'(x) = 2^x \cdot \ln 2 + \frac{1}{x \ln 3} + \sec^2 x - \csc x \cot x$

$f'(x) =$ $2^x \ln 2 + \frac{1}{x \ln 3} + \sec^2 x - \csc x \cot x$

8. [8 points] A particle is moving in a straight line so that its displacement from the origin is given by $s(t) = t^4 - 4t^3 + 4t^2 + 2$. If $t \geq 0$ represents time since the particle started moving, when is the particle's displacement from the origin increasing?

   (Express your final answer using interval notation.)

$$S(t) = t^4 - 4t^3 + 4t^2 + 2$$

$$S'(t) = 4t^3 - 12t + 8t \overset{\text{set}}{=} 0$$

$$4t(t^2 - 3t + 2) = 0$$

$$4t(t-1)(t-2) = 0$$

$$t = 0, 1, 2$$

Make Sign Chart

Test Point	$\frac{1}{2}$	$\frac{3}{2}$	1000000
Sign of s'	+	−	+
Conclusion	Inc	Dec	Inc

Interval of Increase: $(0,1) \cup (2, \infty)$

Answer: $(0, 1) \cup (2, \infty)$

9. [8 points] For what values of x is the function $f(x) = e^{-\frac{x^2}{2}}$ concave downward. (Express your final answer using interval notation.)

$$f(x) = e^{-\frac{x^2}{2}}$$

$$f'(x) = -xe^{-\frac{x^2}{2}}$$

$$f''(x) = -e^{-\frac{x^3}{2}} + (-x)\left(-xe^{-\frac{x^3}{2}}\right)$$

$$= (x^2-1)\underbrace{e^{-\frac{x^3}{2}}}_{\text{Never } 0} \overset{\text{set}}{=} 0$$

$$x^2 - 1 = 0$$

$$x = \pm 1$$

Make Sign Chart for Concavity

-1 $\quad$ 1

Test Point	-1000000	0	1000000
Sign of f''	$+$	$-$	$+$
Conclusion	Con. Up	Con. Down.	Con. Up

Therefore, f is CD on $(-1,1)$

Answer: $(-1, 1)$

10. Consider the function $f(x) = x^6 + x^4$.

(a) [4 points] Show $f''(0) = 0$.

Solution: $f''(x) = 30x^4 + 12x^2 = 6x^2(5x^2 + 2)$, so $f''(0)$ is clearly 0.

(No answer box for Part (a).)

(b) [4 points] Does f have an inflection point at $x = 0$? Explain why or why not.

Analyze the concavity near 0

Test Point	-1	1
Sign of f''	$+$	$+$
Conclusion	CU	CU

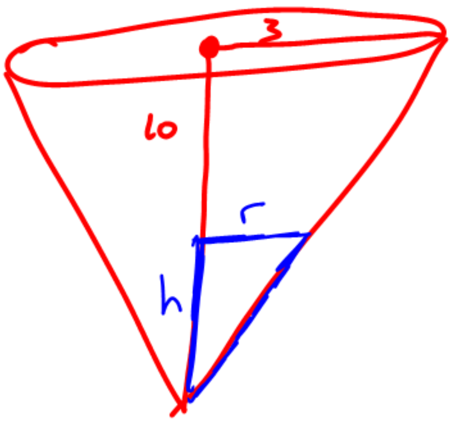
Concavity does not change at 0, so it is not an inflection point.

No, since f does not change concavity at $x = 0$.

11. [8 points] A paper cup has the shape of a cone with height 10 cm and radius 3 cm (at the top). If water is poured into the cup at a rate of $2\,\text{cm}^3/\text{s}$, how fast is the water level rising when the water is 5 cm deep?

For full points, include the unit in your final answer.

(The volume of a cone with a base radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)



We need to find $\frac{dh}{dt}$ when $h=5$.
We write the Volume as a fcn of height

$$V = \frac{1}{3}\pi r^2 h$$

We also know $\frac{r}{h} = \frac{3}{10} \Rightarrow r = \frac{3}{10}h$

$$V(h) = \frac{1}{3}\pi\left(\frac{3}{10}h\right)^2 h$$

$$V(h) = \frac{3\pi}{100}h^3$$

$$\frac{dV}{dt} = \frac{3\pi}{100}\left(3h^2\frac{dh}{dt}\right)$$

$$\frac{dh}{dt} = \frac{100}{9\pi h^2}\frac{dV}{dt}$$

Thus $\left.\frac{dh}{dt}\right]_{\substack{h=5\\V'=2}} = \frac{100}{9\pi(5)^2}(2) = \frac{8}{9\pi}\frac{\text{cm}}{\text{s}}$

Answer: $\frac{8}{9\pi}$ cm/s

12. [8 points] Find $\frac{dy}{dx}$ at the point $P(0, \frac{\pi}{2})$, where the equation:

$$x^2 \cos y + \sin 2y = xy$$

implicitly defines y as a differentiable function of x.

$$x^2 \cos y + \sin 2y = xy$$

$$\frac{d}{dx}(x^2 \cos y + \sin 2y) = \frac{d}{dx}(xy)$$

$$2x \cos y - x^2[\sin y]\frac{dy}{dx} + 2[\cos 2y]\frac{dy}{dx} = y + x\frac{dy}{dx}$$

$$2[\cos 2y]\frac{dy}{dx} - x^2[\sin y]\frac{dy}{dx} - x\frac{dy}{dx} = y - 2x\cos y$$

$$\frac{dy}{dx}\left[2\cos 2y - x^2 \sin y - x\right] = y - 2x\cos y$$

$$\frac{dy}{dx} = \frac{y - 2x\cos y}{2\cos 2y - x^2 \sin y - x}$$

$$\left.\frac{dy}{dx}\right]_{(0, \frac{\pi}{2})} = \frac{\frac{\pi}{2} - 0}{2(-1) - 0 - 0} = -\frac{\pi}{4}$$

$\left.\frac{dy}{dx}\right|_{(0,\ \frac{\pi}{2})} = -\frac{\pi}{4}$