MA 161 ONLINE, SUMMER 2026
FINAL EXAM

Name (Print): Key ____________________ PUID: ____________________

Section Number: ____________________ Name of TA: ____________________

Start Time: ____________________ End Time: ____________________

Directions:

- Please use a **pen** if possible. Marks made by pen scan better than marks made by pencil.
- This is a **two-hour** exam.
- There are **24 questions** worth 8 points each; the maximum score is 200 points.
- The remaining 8 points are earned for filling out the information at the top of the page correctly. Section numbers, and their associated TA, are provided in the table below.

203	June Marie Weiland	**204**	Laura Maria Castillo

- Circle one and only one choice for each multiple-choice problem (Problems 1-6). **No partial credit** will be given on multiple-choice problems.
- Show all relevant work on each free-response problem (Problems 7-24). You must include some work or justification to receive full credit on these problems; partial credit may be given for steps leading to the correct solution. **Write the final answer in the box(es) provided**.
- Any external help or communication is not allowed, except with the proctor for logistical reasons. This exam is closed book and closed notes. **Calculators are not allowed**. Electronic devices, including smart watches, are to be turned off and put away.
- The last sheet is scratch paper. You may use only the scratch paper provided in this exam booklet. No additional scratch paper is permitted.

Purdue University faculty and students commit themselves towards maintaining a culture of academic integrity and honesty. The students taking this exam are not allowed to seek or obtain any kind of help from anyone to answer questions on this test. If you have questions, consult only an instructor or a proctor. You are not allowed to look at the exam of another student. You may not compare answers with anyone else or consult another student until after you finish your exam and hand it in to a proctor or to an instructor. You may not consult notes, books, calculators, cameras, or any kind of communications devices until after you finish your exam and hand it in to a proctor or to an instructor. If you violate these instructions you will have committed an act of academic dishonesty. Penalties for academic dishonesty can be very severe and may include an F in the course. All cases of academic dishonesty will be reported to the Office of the Dean of Students. Your instructor and proctors will do everything they can to stop and prevent academic dishonesty during this exam. If you see someone breaking these rules during the exam, please report it to the proctor or to your instructor immediately. Reports after the fact are not very helpful.

I agree to abide by the instructions above:

Student Signature: ____________________

Proctor Signature: ____________________

Question	Points	Score
1	8	
2	8	
3	8	
4	8	
5	8	
6	8	
7	8	
8	8	
9	8	
10	8	
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18	8	
19	8	
20	8	
21	8	
22	8	
23	8	
24	8	
Correct Header Info	8	
Total:	200	

1. [8 points] Simplify $\sec(\tan^{-1} x)$.

   A. x

   B. $\sqrt{x^2+1}$

   C. $\frac{x}{\sqrt{x^2+1}}$

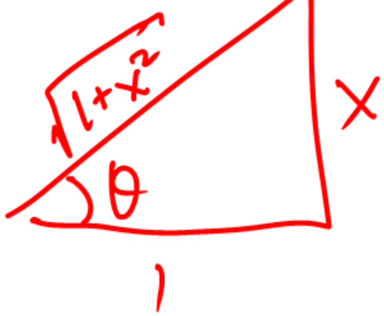
   D. $\frac{1}{\sqrt{x^2+1}}$

   E. $\frac{\sqrt{x^2+1}}{x}$

Let $\theta = \tan^{-1} x$

$\frac{x}{1} = \tan\theta$

Make Right Triangle



$\sec \tan^{-1} x = \sec\theta = \frac{\text{hyp}}{\text{adj}}$

$= \frac{\sqrt{x^2+1}}{1} = \sqrt{x^2+1}$

2. [8 points] When is $f(x)$ discontinuous?

$$f(x) = \begin{cases} \frac{\sin x}{x} & x < 0 \\ \frac{1}{1-x} & 0 \leq x \leq 2 \\ e^{x-2} - 1 & x > 2 \end{cases}$$

A. $x = 1$

B. $x = 2$

C. $x = 0$ and $x = 2$

D. $x = 1$ and $x = 2$

E. f is continuous everywhere.

Candidates for discontinuities : $x = 0, 1, 2$

f is continuous at 0:

$$\lim_{x \to 0^-} f = \lim_{x \to 0^-} \frac{\sin(x)}{x} = 1 = f(1) = \lim_{x \to 0^+} f$$

f has a vertical asymptote at $x = 1$

f has a jump discontinuity at $x = 2$:

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} \left(e^{x-2} - 1\right) = 1 - 1 = 0$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{1}{1-x} = -1$$

3. [8 points] Let $F(x) = \int_0^x f(t)\,dt$. Which of the following statements are TRUE:

(i) $F(x)$ is increasing when $f(x) > 0$;

(ii) $F(x)$ has a critical point when $f(x) = 0$;

(iii) $F(x)$ is concave upward when $f(x)$ is decreasing

A. (ii) only

B. (i) and (ii) only

C. (i) and (iii) only

D. (i), (ii), and (iii)

E. None of them are true

By FTC, $F'(x) = f(x)$
$F''(x) = f'(x)$

(i) is true
$f(x) > 0 \implies F'(x) > 0$
$\implies F$ is increasing

(ii) is true
$f(x) = 0 \implies F'(x) = 0$
$\implies F$ has a critical point at x

(iii) is false
$f(x)$ decreasing $\implies f'(x) < 0$
$\implies F''(x) < 0$
$\implies F$ is concave downward
$\implies F$ is NOT concave upward

4. [8 points] Let M and m be the maximum and minimum of $f(x) = x^3 - 3x^2 + 3$ on the interval $[1, 4]$, respectively. What is the value of $M + m$?

   A. 0
   B. 10
   C. 18
   D. 20
   E. 22

$f'(x) = 3x^2 - 6x$

$= 3x(x-2) \overset{\text{set}}{=} 0$

$x = 0, 2$

$\xrightarrow{x \in [1,4]} x = 2$

Check Values:

x	1	2	4
$f(x)$	1	-1	19

$x^3 - 3x^2 + 3$

$64 - 48 = 16$

$16 + 3 = 19$

$M = 19$

$m = -1$

$M + m = 18$

5. [8 points] Let $f(x)$ be a function such that $\int_2^9 f(x)\,dx = 13$ and $\int_4^9 f(x)\,dx = 7$. Find the value of:

$$\int_4^2 3f(x)\,dx$$

A. -18

B. -6

C. 6

D. 8

E. 18

$$\int_4^2 3f(x)\,dx = -3\left[-\int_4^2 f(x)\,dx\right]$$

$$= -3\int_2^4 f(x)\,dx$$

$$= -3\left[\int_2^9 f(x)\,dx - \int_4^9 f(x)\,dx\right]$$

$$= -3[13-7]$$

$$= -3\cdot 6$$

$$= -18 \checkmark$$

6. [8 points] If $f(x) = \sin^{-1}\left(\frac{3}{5}x^5\right)$, what is $f'(1)$?

   A. $\frac{15}{4}$

   B. $\frac{5}{4}$

   C. $\frac{75}{18}$

   D. $-3\csc\left(\frac{3}{5}\right)\cot\left(\frac{3}{5}\right)$

   E. f is not differentiable at $x = 1$

$$f'(x) = \frac{1}{\sqrt{1-\left(\frac{3}{5}x^5\right)^2}} \cdot 3x^4$$

$$f'(1) = \frac{1}{\sqrt{1-\frac{9}{25}}} \cdot 3 = \frac{3}{\sqrt{\frac{16}{25}}} = \frac{3}{\frac{4}{5}} = \frac{15}{4}$$

7. [8 points] A sample of a radioactive element initially has a mass of 24 g. After 3 minutes, the sample of that element has 2 g. At what time (in minutes after the initial time) is the mass equal to 6 g?

(Keep your answer exact. Do not round.)

This follows an exponential decay model

$$P(t) = P(0)\, e^{kt}$$

Find k:

$$2 = 24 e^{3k}$$

$$\frac{1}{12} = e^{3k}$$

$$\ln \frac{1}{12} = 3k$$

$$k = -\frac{1}{3}\ln 12$$

Solve the problem:

$$P(t) = 24 e^{-\frac{1}{3}\ln 12\, t} \overset{\text{set}}{=} 6$$

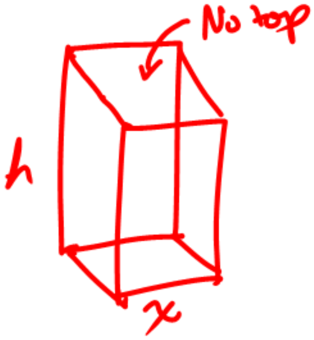
$$e^{-\frac{1}{3}\ln 12\, t} = \frac{1}{4}$$

$$-\frac{1}{3}\ln 12\, t = -\ln 4$$

$$t = \frac{3\ln 4}{\ln 12}$$

Time: $\frac{3\ln 4}{\ln 12}$ minutes

8. [8 points] If $1200\,\text{cm}^2$ of material is available to make a box with a square base and an open top, what are the dimensions of the box with the largest possible volume?



maximize $V = x^2 h$

subject to $1200 = x^2 + 4xh$

$$h = \frac{1200 - x^2}{4x}$$

$$h = \frac{300}{x} - \frac{1}{4}x$$

$$V(x) = x^2\left(\frac{300}{x} - \frac{1}{4}x\right) = 300x - \frac{1}{4}x^3$$

$$V'(x) = 300 - \frac{3}{4}x^2 \overset{\text{set}}{=} 0$$

$$\frac{3}{4}x^2 = 300$$

$$x^2 = 400 \; ; x > 0$$

$$x = 20 \text{ cm}$$

Solve for height:

$$h\Big|_{x=20} = \frac{300}{20} - \frac{1}{4}(20) = 15 - 5 = 10 \text{ cm}$$

Dimensions: $20\text{ cm} \times 20\text{ cm} \times 10\text{ cm}$

Dimensions: The base width is $20\,\text{cm}$, while the height is $10\,\text{cm}$.

9. [8 points] Determine the limit, or explain why it does not exist:

$$\lim_{x\to 1^+} \frac{x+2}{\sqrt{x^2-1}}$$

VA at $x=1$, so it is either ∞ or $-\infty$

Sign of $x+2$ $+$

Sign of $\sqrt{x^2-1}$ $+$

So, $\frac{x+2}{\sqrt{x^2-1}}$ remains positive as $x \to 1^+$, so the limit is ∞.

Answer: $+\infty$

10. [8 points] Consider the function $f(x) = e^{2x}$:

(a) Find a linear approximation $L(x)$ of $f(x)$ at $a = 0$.

$f(0) = 1$

$f'(x) = 2e^{2x}$

$f'(0) = 2$

So,

$L(x) = f(0) + f'(0)(x-0)$

$= 1 + 2x$

$L(x) = 1 + 2x$

(b) Use your answer in (a) to (very crudely) approximate the value of e.

$e^{2x} \approx L(x)$ near 0. So,

$e = e^{2(\frac{1}{2})} \approx L(\frac{1}{2}) = 1 + 2(\frac{1}{2}) = 1 + 1 = 2$

For context, $e \approx 2.71828....$, so this is a very bad approximation. You'll learn a better way in MA 162.

$e \approx 2$

11. [8 points] Let ℓ be the line tangent to the curve $g(x) = (2x-1)(x^2+4x+4)$ at $x = 1$. Where does ℓ intersect the $x-$axis?

$g(1) = 1 \cdot 9 = 9$

$g'(x) = 2(x^2+4x+4) + (2x-1)(2x+4)$

$g'(1) = 2(9) + 1 \cdot 6$

$= 24$

So ℓ: $y - 9 = 24(x-1)$

$y = 9 + 24x - 24$

$y = 24x - 15$

x-intercept: $0 = 24x - 15$

$x = \frac{15}{24}$

$x = \frac{5}{8}$

$x = \frac{5}{8}$

12. [8 points] Evaluate the integral:

$$\int_0^{\frac{1}{3}} \frac{1}{1+(3x)^2}\, dx$$

Let $u = 3x$, then

$du = 3\,dx \Rightarrow dx = \frac{1}{3} du$

$x = 0 \Rightarrow u = 0$

$x = \frac{1}{3} \Rightarrow u = 1$

So, $\int_0^{1/3} \frac{1}{1+(3x)^2}\, dx = \int_0^1 \frac{1}{1+u^2} \frac{1}{3} du = \frac{1}{3}\int_0^1 \frac{1}{1+u^2}\, du$

$= \frac{1}{3}\left[\tan^{-1} u\right]_0^1$

$= \frac{1}{3}\left[\tan^{-1} 1 - \tan^{-1} 0\right]$

$= \frac{1}{3} \cdot \frac{\pi}{4}$

$= \frac{\pi}{12}$

$$\int_0^{\frac{1}{3}} \frac{1}{1+(3x)^2}\, dx = \frac{\pi}{12}$$

13. [8 points] Evaluate the limit below using any valid method. Make sure to show your work.

$$\lim_{x\to 0} \frac{\sin 3x}{\tan 5x}$$

Let $\underline{\underline{L'H}}$ denote the use of L'Hôpital's Rule

$$\lim_{x\to 0} \frac{\overset{\nearrow 0}{\sin 3x}}{\underset{\searrow 0}{\tan 5x}} \overset{L'H}{=\!=} \lim_{x\to 0} \frac{3\cos 3x}{5\sec^2 5x} = \frac{3}{5} \lim_{x\to 0} \cos 3x \cos^2 5x$$

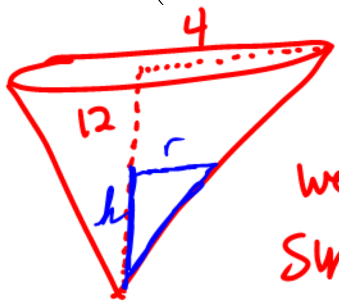
$$= \frac{3}{5} \cdot 1 \cdot 1^2 = \frac{3}{5}$$

$$\boxed{\lim_{x\to 0} \frac{\sin 3x}{\tan 5x} = \frac{3}{5}}$$

14. [8 points] A paper cup has the shape of a cone with height 12 cm and radius 4 cm (at the top). If water is poured into the cup at a rate of $2\,\text{cm}^3/\text{s}$, how fast is the water level rising when the water is 6 cm deep?

For full points, include the unit in your final answer.

$\left(\text{The volume of a cone with a base radius } r \text{ and height } h \text{ is } V = \frac{1}{3}\pi r^2 h.\right)$



$V = \frac{1}{3}\pi r^2 h$, to make our lives easier we write r as a function of h. By similar triangles,

$$\frac{r}{h} = \frac{4}{12} \Rightarrow r = \frac{1}{3}h$$

So, $$V(h) = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h = \frac{\pi}{27}h^3$$

$$\frac{dV}{dt} = \frac{\pi}{9}h^2\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{9}{\pi h^2}\frac{dV}{dt}$$

$$\left.\frac{dh}{dt}\right]_{\substack{h=6\\ V'=2}} = \frac{9}{\pi(36)}(2) = \frac{18}{\pi(36)} = \frac{1}{2\pi}\ \frac{\text{cm}}{\text{s}}$$

Answer: $\frac{1}{2\pi}\text{cm/s}$

15. [8 points] Find a function $f(x)$ such that $f'(x) = 2x + 3\sin x$ and $f(0) = 5$.

$f(x) = \int f'(x)\,dx = x^2 - 3\cos x + C$ for some C.

$f(0) = 0 - 3\cdot 1 + C \overset{\text{set}}{=} 5$

$C = 8$

$\Rightarrow f(x) = x^2 - 3\cos x + 8$

$f(x) = x^2 - 3\cos x + 8$

16. [8 points] Evaluate the integral:

$$\int \frac{2x^5 - \sqrt{x}}{x}\, dx$$

(Don't forget your constant of integration. You will lose points if you do.)

$$\int \frac{2x^5 - \sqrt{x}}{x}\, dx = \int \left(\frac{2x^5}{x} - \frac{\sqrt{x}}{x}\right) dx = \int \left(2x^4 - x^{-\frac{1}{2}}\right) dx$$

$$= \frac{2}{5}x^5 - 2x^{\frac{1}{2}} + C$$

$$\boxed{\int \frac{2x^5 - \sqrt{x}}{x}\, dx = \frac{2}{5}x^5 - 2\sqrt{x} + C}$$

17. [8 points] Find the initial acceleration of a particle if its position after t seconds is $s(t) = \ln(t^2 + 1)$. Express your final answer in m/s^2

$$s(t) = \ln(t^2+1)$$

$$v(t) = \frac{2t}{t^2+1}$$

$$a(t) = \frac{2(t^2+1) - 2t(2t)}{(t^2+1)^2}$$

$$a(0) = \frac{2-0}{1^2} = 2 \text{ m/s}^2$$

$a(0) =$ $2\,\mathrm{m/s^2}$

18. Consider the function $f(x) = \ln(x^2)$ on the interval $[-5, 5]$.

(a) [6 points] Show there is **NOT** a point c between -5 and 5 such that $f'(c)$ is the average rate of change of f on $[-5, 5]$.

$$f'(x) = \frac{2x}{x^2} = \frac{2}{x}$$

$$f'(c) = \frac{f(5) - f(-5)}{5-(-5)}$$

$$\frac{2}{c} = \frac{\ln 25 - \ln 25}{10}$$

$$\frac{2}{c} = 0$$

No Solution

(No answer box for Part (a).)

(b) [2 points] Which condition(s) of the Mean Value Theorem are not satisfied by $\ln(x^2)$ on the interval $[-5, 5]$?

f is not differentiable at $x = 0$. Hence, it is not differentiable on $(-5, 5)$, nor continuous on $[-5, 5]$.

19. [8 points] For what values of x is the function $f(x) = xe^{-x}$ concave upward?
(Express your final answer using interval notation.)

$$f'(x) = e^{-x} - x e^{-x} = (1-x)e^{-x}$$

$$f''(x) = -e^{-x} - (1-x)e^{-x}$$

$$= e^{-x}(-1-1+x)$$

$$= e^{-x}(x-2) \overset{\text{set}}{=} 0$$

$$x = 2$$

Sign Chart

	$(-\infty, 2)$	$(2, \infty)$
Sign of e^{-x}	$+$	$+$
Sign of $x-2$	$-$	$+$
Sign of f''	$-$	$+$
Result	CD	CU

f is con cave down ward on $(2, \infty)$

Answer: $(2, \infty)$

20. [8 points] Find $\frac{dy}{dx}$, where the equation:

$$xy - \cos y = x^2 + y^2$$

implicitly defines y as a differentiable function of x.
(You do **NOT** need to write $\frac{dy}{dx}$ as an expression of only x.)

$$y + x\frac{dy}{dx} + \sin y\frac{dy}{dx} = 2x + 2y\frac{dy}{dx}$$

$$x\frac{dy}{dx} + \sin y\frac{dy}{dx} - 2y\frac{dy}{dx} = 2x - y$$

$$\frac{dy}{dx}(x + \sin y - 2y) = 2x - y$$

$$\frac{dy}{dx} = \frac{2x - y}{x + \sin y - 2y}$$

$$\frac{dy}{dx} = \frac{2x - y}{x + \sin y - 2y}$$

21. [8 points] A polynomial $f(x)$ has a derivative $f'(x) = (x+2)^3(x-1)^2(x-4)(x-5)$. For what values of x is $f(x)$ decreasing?

    (Express your final answer using interval notation.)

Critical points $x = -2, 1, 4, 5$

Sign Chart

	$(-\infty,-2)$	$(-2,1)$	$(1,4)$	$(4,5)$	$(5,\infty)$
Sign of $(x+2)^3$	$-$	$+$	$+$	$+$	$+$
Sign of $(x-1)^2$	$+$	$+$	$+$	$+$	$+$
Sign of $x-4$	$-$	$-$	$-$	$+$	$+$
Sign of $x-5$	$-$	$-$	$-$	$-$	$+$
Sign of f'	$-$	$+$	$+$	$-$	$+$
Results	Dec	Inc	Inc	Dec	Inc

Interval of Decrease: $(-\infty,-2) \cup (4,5)$

Answer: $(-\infty,-2) \cup (4,5)$

22. [8 points] Determine all vertical and horizontal asymptotes for the function below:

$$r(t) = \frac{5t(t+3)(t-1)}{t(3t-6)(t+1)}$$

VAs: Either $t=0$ OR $3t-6=0$ OR $t+1=0$

$\boxed{t=2}$ $\boxed{t=-1}$

Numerator is also 0, r has a hole at 0 not VA

HAs:

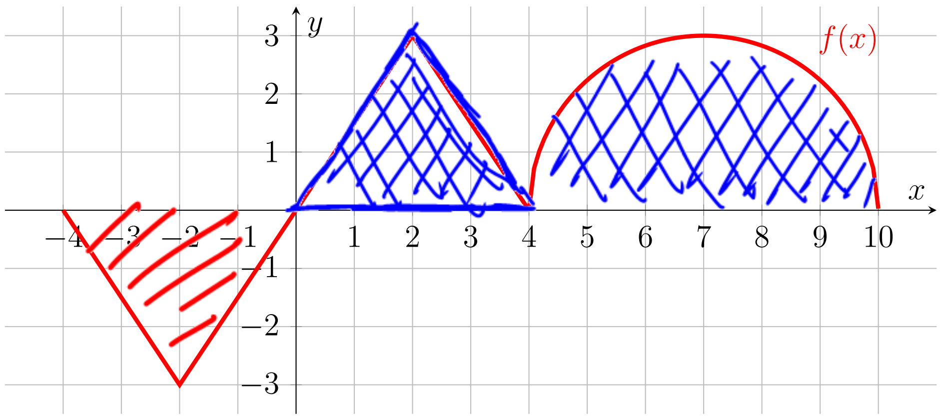
$$r(t) = \frac{5t^3 + [\text{Irrelevant when } t \text{ large}]}{3t^3 + [\text{Irrelevant when } t \text{ large}]} \xrightarrow{t \to \pm\infty} \frac{5}{3}$$

The only HA is $y = \frac{5}{3}$

Vertical Asymptote(s): $t = -1, 2$

Horizontal Asymptote(s): $y = \frac{5}{3}$

23. [8 points] Compute $\int_{-4}^{10} f(x)\,dx$, where the graph of $f(x)$ is depicted below.



$$\int_{-4}^{10} f(x)\,dx = \text{Area of Blue Triangle} + \text{Area of Semi circle} - \text{Area of Red Triangle}$$

$$= \frac{1}{2}(4)(3) + \frac{1}{2}\pi(3)^2 - \frac{1}{2}(4)(3)$$

$$= 6 + \frac{9}{2}\pi - 6$$

$$= \frac{9}{2}\pi$$

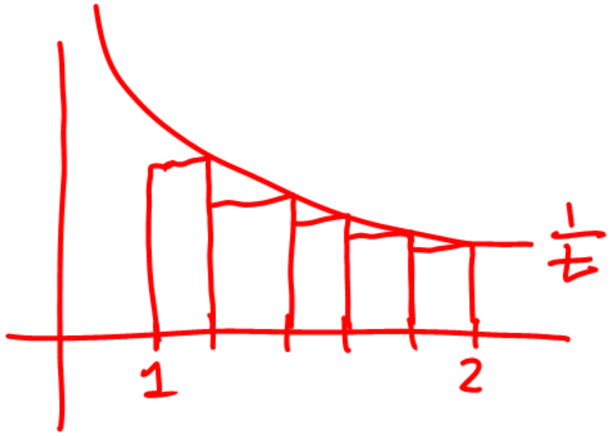
$$\boxed{\int_{-4}^{10} f(x)\,dx = \frac{9}{2}\pi}$$

24. [8 points] One can show:

$$\ln 2 = \int_1^2 \frac{1}{t}\, dt$$

Set up a right Riemann sum with N rectangles that approximates the value of $\ln 2$.

Simplify your answer, **but do NOT evaluate the sum.**



$f(t) = \frac{1}{t}$

$a = 1$

$b = 2$

$\Delta t = \frac{2-1}{N} = \frac{1}{N}$

$$\ln 2 = \int_1^2 \frac{1}{t}\, dt \approx R_N = \sum_{i=1}^{N} f(a + i\Delta t)\,\Delta t$$

$$= \sum_{i=1}^{N} \frac{1}{1 + i\left(\frac{1}{N}\right)} \cdot \frac{1}{N}$$

$$= \sum_{i=1}^{N} \frac{1}{N + i}$$

$$\ln 2 \approx \frac{1}{N} \sum_{i=1}^{N} \frac{1}{1 + \frac{i}{N}} = \sum_{i=1}^{N} \frac{1}{N+i}$$