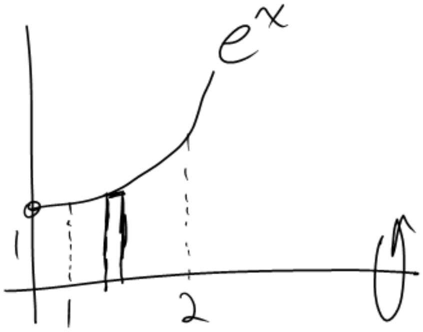
Final Exam Review

This Friday 8/8, 1–3 PM, WALC B066

Volumes of Revolution:



$f(x) = e^x$

Disk/Washer Method

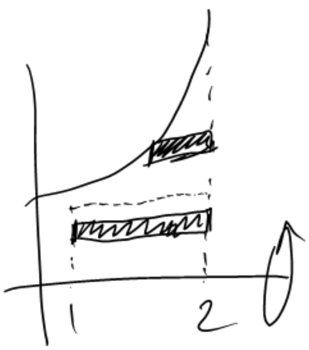
$$\int \pi [f(x)]^2 \, dx$$

Shell Method

$$\int 2\pi x \, f(x) \, dx$$

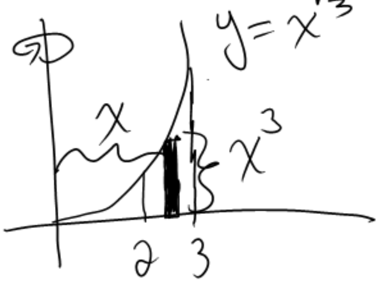
Disk Method: $\int_1^2 \pi [e^x]^2 \, dx = \frac{\pi}{2} \int_1^2 2e^{2x} \, dx = \frac{\pi}{2} [e^{2x}]_1^2$

$= \frac{\pi}{2} [e^4 - e^2]$



$$\int_{y=0}^{e} 2\pi y [2-1] \, dy + \int_{e}^{e^2} 2\pi y [2 - \ln y] \, dy$$

Another Example



Formula: $\int 2\pi x f(x) \, dx$

$\int_2^3 2\pi x [x^3] \, dx = \int_2^3 2\pi x^4 \, dx$

$= 2\pi \cdot \frac{1}{5} [x^5]_2^3 = \frac{2\pi}{5} [3^5 - 2^5]$

$\int (\ln y) \, y \, dy = (\ln y)(\frac{1}{2} y^2) - \int \frac{1}{2} y^2 \cdot \frac{1}{y} \, dy$

$u = \ln y \quad v = \frac{1}{2} y^2$

$du = \frac{1}{y} dy \quad dv = y \, dy$

$= (\ln y)(\frac{1}{2} y^2) - \frac{1}{2} \int y \, dy$

$= (\ln y)(\frac{1}{2} y^2) - \frac{1}{2} \cdot \frac{1}{2} y^2 + C$

Arc Length:

In Cartesian, $\int_a^b ds$; $ds = \sqrt{1+[f'(x)]^2}\,dx$

$y = f(x)$

$= \int_a^b \sqrt{1+[f'(x)]^2}\,dx$; $a \le x \le b$

In polar, $\int_a^b ds = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$; $r = f(\theta)$, $a \le \theta \le b$

Ex/ Compute the length of the curve for $y = x^2$ from $0 \le x \le 1$.

$y = x^2$

$\frac{dy}{dx} = 2x$

$\mathcal{L} = \int_0^1 ds = \int_0^1 \sqrt{1+(2x)^2}\,dx = \int_0^1 \sqrt{1+4x^2}\,dx$

$\tan^2 + 1 = \sec^2$

$\int \sqrt{1+(2x)^2}\,dx = \int \sec\theta \left(\tfrac{1}{2}\sec^2\theta\right) d\theta = \frac{1}{2}\int \underbrace{\sec\theta}_{u}\, \underbrace{\sec^2\theta\, d\theta}_{dv}$

$\frac{2x}{1} = \tan\theta$

$x = \frac{1}{2}\tan\theta$

$dx = \frac{1}{2}\sec^2\theta\, d\theta$

$\sec\theta = \frac{\sqrt{1+(2x)^2}}{1}$

$\sec\theta\tan\theta - \int \tan^2\theta \sec\theta\, d\theta$

$= \sec\theta\tan\theta - \int (\sec^2\theta - 1)\sec\theta\, d\theta$

$= \sec\theta\tan\theta - \int \sec^3\theta\, d\theta + \int \sec\theta\, d\theta$

$2\int \sec^3\theta\, d\theta = \sec\theta\tan\theta + \int \sec\theta\, d\theta$

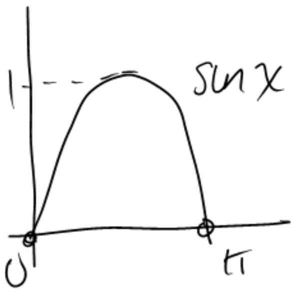
$\int \sec^3\theta\, d\theta = \frac{1}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right] + C_0$

$$\int\sqrt{1+4x^2}\,dx = \frac{1}{2}\left[2x\sqrt{1+(2x)^2} + \ln\left(\sqrt{1+(2x)^2}+2x\right)\right] + C$$

Finally,

$$\mathcal{L} = \int_0^1 \sqrt{1+4x^2}\,dx = \underset{x=1}{\text{This at}} = \sqrt{5} + \frac{1}{2}\ln\left(\sqrt{5}+2\right)$$

Ex/ Determine the surface area of the region obtained when $f(x) = \sin x$ $[0 \leq x \leq \pi]$ is revolved around the x-axis.



Surface Area: $\int_a^b 2\pi y\,ds = \int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$y = \sin x$; $\left(\frac{dy}{dx}\right)^2 = \cos^2 x$

$\frac{dy}{dx} = \cos x$

$$S = \int_0^{\pi} 2\pi[\sin x]\sqrt{1+\cos^2 x}\,dx$$

Ex/ Find the length of the polar curve $r = \sqrt{1+\sin 2\theta}$ from $0 \leq \theta \leq \frac{\pi}{\sqrt{2}}$

Formula: $\mathcal{L} = \int_a^b ds = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$

$r^2 = 1+\sin 2\theta$

$2r\frac{dr}{d\theta} = 2\cos 2\theta \Rightarrow \frac{dr}{d\theta} = \frac{\cos 2\theta}{r} \Rightarrow \left(\frac{dr}{d\theta}\right)^2 = \frac{\cos^2 2\theta}{r^2}$

$= \frac{\cos^2 2\theta}{1+\sin 2\theta}$

$$\mathcal{L} = \int_0^{\pi/\sqrt{2}} \sqrt{(1+\sin 2\theta) + \frac{\cos^2 2\theta}{1+\sin 2\theta}}\,d\theta = \int_0^{\pi/\sqrt{2}} \sqrt{\frac{1+2\sin 2\theta + \sin^2 2\theta + \cos^2 2\theta}{1+\sin 2\theta}}\,d\theta = \int_0^{\pi/\sqrt{2}} \sqrt{2}\,d\theta = \pi$$

Power Series, Polar Coordinates

§11.1 – §11.4 (Power Series)

$\sum_{n=0}^{\infty} C_n (x-a)^n$. There are 3 possibilities for convergence

1. Only converges at a
2. For all $x \in \mathbb{R}$
3. Converges only a certain distance away from a.

Ex/ Determine the radius and interval of convergence for

$$\sum_{n=1}^{\infty} \frac{(2x)^n}{\sqrt{n}}$$

Use Ratio Test

$$\lim_{n\to\infty} \left| \frac{(2x)^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(2x)^n} \right| = \lim_{n\to\infty} \left| (2x)^{n+1-n} \cdot \frac{\sqrt{n}}{\sqrt{n+1}} \right| = \lim_{n\to\infty} |2x| \cdot \sqrt{\frac{n}{n+1}}$$

$$= |2x| \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = |2x| \cdot 1 = |2x| < 1 \Rightarrow |x| < \frac{1}{2}$$

Radius: $\frac{1}{2}$

Will converge on $(-\frac{1}{2}, \frac{1}{2})$, now check endpoints

When $x = -\frac{1}{2}$: $\sum_{n=1}^{\infty} \frac{(2 \cdot -\frac{1}{2})^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ Check AST

$\frac{1}{\sqrt{n}}$ positive and decreasing

$\frac{1}{\sqrt{n}} \to 0$ as $n \to \infty$

When $x = \frac{1}{2}$: $\sum_{n=1}^{\infty} \frac{(2 \cdot \frac{1}{2})^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ divergent p-series ($p = \frac{1}{2}$)

Therefore, the interval is $[-\frac{1}{2}, \frac{1}{2})$

Term-by-Term Differentiation and Integration:

When x is within the radius of convergence

$$\frac{d}{dx}\sum_{n=0}^{\infty} C_n (x-a)^n = \sum_{n=0}^{\infty} C_n \frac{d}{dx}(x-a)^n$$

$$\int \sum_{n=0}^{\infty} C_n (x-a)^n dx = \sum_{n=0}^{\infty} C_n \int (x-a)^n dx$$

Quiz 9 #2 $\quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$; $x \in (-\infty, \infty)$

(1) Determine a power series rep. for $x^2 e^x$

$$x^2 e^x = x^2 \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^2 \cdot x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!}$$

(2) Determine " " " " for e^{-x^2}

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

Suppose its interval of convergence is $(-\infty, \infty)$

(3) Find an anti-derivative for e^{-x^2}

$$\int e^{-x^2} dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int x^{2n} dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{x^{2n+1}}{2n+1} + C$$

(4) Show that the error of approximating e^{-x^2} by the N^{th} term of the taylor series goes to 0 as $N \to \infty$

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}. \quad \text{Error} = |R_N|$$

$$|R_N| \leqslant \left| \frac{(-1)^{N+1}}{(N+1)!} x^{2(N+1)} \right| \leqslant \frac{|x|^{2N+2}}{(N+1)!} \longrightarrow 0$$

Taylor Series: For a function f, $\sum_{n=0}^{\infty} C_n (x-a)^n$ where

① $C_n = \frac{f^{(n)}(a)}{n!}$

② $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ when $|R_N| \to 0$

Ex The taylor series for $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, Compute $f^{(2025)}(0)$

When $|x|<1$,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

So, $\frac{f^{(n)}(0)}{n!} = 1$ for all n, when $n = 2025$

$$\frac{f^{(2025)}(0)}{2025!} = 1 \Rightarrow f^{(2025)}(0) = 2025!$$

Another way Recognize $f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$

Quiz 10 #1 Prove $\lim_{x \to 0} \frac{\sin x}{x} = 1$

$$\frac{1}{x}\sin x = \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$$

$$1 + \begin{bmatrix} \text{something that} \\ \text{goes to } 0 \text{ as } x \to 0 \end{bmatrix} \longrightarrow 1 \quad \text{as } x \to 0$$

Polar Coordinates

Conversion Formulas: Cartesian $\longrightarrow$ Polar

$$(x,y) \longmapsto (r,\theta)$$

$$r^2 = x^2 + y^2$$

$$\tan\theta = \frac{y}{x}$$

Polar $\longrightarrow$ Cartesian

$$(r,\theta) \longmapsto (x,y)$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Ex/ Convert the graph $y = x^2$ into Polar

$$r\sin\theta = (r\cos\theta)^2$$

$$r\sin\theta = r^2\cos^2\theta$$

Quiz 10 #3 Describe the curve $r = 6\sin\theta$

$$r^2 = 6r\sin\theta$$

$$x^2 + y^2 = 6y$$

$$x^2 + y^2 - 6y + 9 = 0 + 9$$

$$x^2 + (y-3)^2 = 9$$

Circle with radius 3 and center $(0,3)$

Ex/ $\frac{1}{r} = 2 \quad (r>0)$

$$\frac{1}{\sqrt{x^2+y^2}} = 2 \Rightarrow \frac{1}{2} = \sqrt{x^2+y^2} \Rightarrow x^2+y^2 = \frac{1}{4} \quad [(x,y) \neq (0,0)]$$

<u>Integration in Polar Coordinates</u> $r = f(\theta) \; ; \alpha \le \theta \le \beta$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

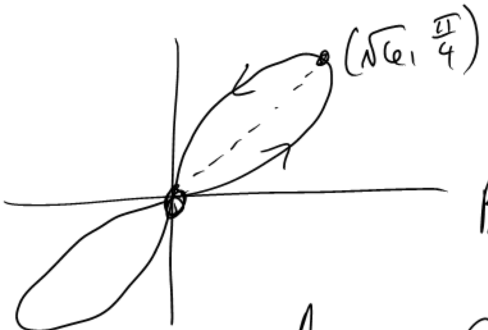
Ex/ Find the region bounded by the lemniscate

$r^2 = 6\sin 2\theta$ <u>Find Bounds</u>: $0 = 6\sin 2\theta$

$\sin 2\theta = 0$

$2\theta = \pi n \; ; n \in \mathbb{Z}$

$\theta = \frac{\pi}{2} n$



Bounds: $[0, \frac{\pi}{2}]$

$$\text{Area} = 2\left[\begin{matrix}\text{Area of}\\ \text{one half}\end{matrix}\right] = 2\int_0^{\pi/2} \frac{1}{2}(6\sin 2\theta)\, d\theta$$

$$= \int_0^{\pi/2} 6\sin 2\theta \, d\theta = 3\int_0^{\pi/2} \sin \underbrace{2\theta}_{u=2\theta} \, (2)\, d\theta$$

$$= 3\Big[-\cos 2\theta\Big]_0^{\pi/2} = 3[1+1] = 6$$

Complete Course Evaluations

Contact me via email if needed, pence11@purdue.edu