MA 16200: Plane Analytic Geometry and Calculus II

Lecture 13: Partial Fractions (Irreducible Quadratic Cases)

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Sections Covered: 8.5

Non-Repeating Linear Terms

Problem 1

Compute $\int \frac{1}{x^2-4}\,dx$ $= \int \frac{1}{(x-2)(x+2)}\,dx =$

$$\left[\frac{1}{(x-2)(x+2)} = \frac{\overbrace{A}^{\text{deg } 0}}{\underbrace{x-2}_{\text{deg } 1}} + \frac{B}{x+2}\right](x-2)(x+2)$$

$$1 = A(x+2) + B(x-2)$$

$$1 = Ax + 2A + Bx - 2B$$

$$0x + 1 \quad (A+B)x + (2A-2B)$$

$$= \begin{cases} A+B=0 \rightarrow B=-A \rightarrow \boxed{B=-1/4} \\ 2A-2B=1 \rightarrow 2A+2A=1 \rightarrow \boxed{A=1/4} \end{cases}$$

$$\int\left(\frac{1/4}{x-2} - \frac{1/4}{x+2}\right)dx$$

$$= \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + c$$

$$= \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C$$

Repeating Linear Terms

Problem 2

Compute $\int \frac{x-1}{x^3+x^2}\,dx$ $= \int \frac{x+1}{x^2(x+1)}\,dx$

$$\left[\frac{x-1}{x^2(x+1)} = \frac{\overbrace{Ax+B}^{\text{deg } 1}}{\underbrace{x^2}_{\text{deg } 2}} + \frac{C}{x+1} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}\right] x^2(x+1)$$

$$x-1 = Ax(x+1) + B(x+1) + Cx^2$$

$$x-1 = Ax^2 + Ax + Bx + B + Cx^2$$

$$x-1 = (A+C)x^2 + (A+B)x + B$$

$$\begin{cases} \boxed{B=-1} \\ A+B=1 \\ A+C=0 \end{cases}$$

$$A-1=1 \longrightarrow \boxed{A=2}$$

$$A+C=0 \longrightarrow \boxed{C=-2}$$

$$\int \frac{x-1}{x^2(x+1)} dx = \int \left(\frac{2}{x} - \frac{1}{x^2} - \frac{2}{x+1}\right) dx$$

$$= 2 \ln|x| + \frac{1}{x} - 2 \ln|x+1| + K$$

* In this section K will refer to the constant of integration

General Strategy

For each factor of the form $(x - r)$ in the denominator, the term in the Partial Fraction Decomposition is:

$$\frac{A}{x - r}$$

For each factor of the form $(x - a)^n$ for $n > 1$,

$$\frac{A_1}{x - a} + \frac{A_2}{(x - a)^2} + \ldots + \frac{A_n}{(x - a)^n}$$

Example

Problem 3

Compute $\int \frac{3x^2+1}{x(x^2+1)}\,dx$

$$\left[\frac{3x^2+1}{x(x^2+1)} = \frac{\overbrace{A}^{\text{deg }0}}{\underbrace{x}_{\text{deg }1}} + \frac{\overbrace{Bx+C}^{\text{deg }1}}{\underbrace{x^2+1}_{\text{deg }2}}\right] x(x^2+1)$$

$$3x^2+1 = A(x^2+1) + (Bx+C)x$$

$$3x^2+1 = Ax^2+A+Bx^2+Cx$$

$$= (A+B)x^2+Cx+A$$

$$\begin{cases} A+B=3 \\ \boxed{C=0} \end{cases}$$

$$\boxed{A=1}$$

$$1+B=3 \rightarrow \boxed{B=2}$$

$$= \int \left(\frac{1}{x} + \frac{2x}{x^2+1}\right) dx = \int \frac{1}{x}\, dx + \int \frac{2x}{\underbrace{x^2+1}_{u=x^2+1}}\, dx$$

$$= \ln|x| + \ln|x^2+1| + K$$

General Strategy

Sometimes it is better to write it as

$$\frac{A(2ax+b)+B}{ax^2+bx+c}$$

For each **irreducible** factor of the form $(ax^2 + bx + c)$ in the denominator, the term in the Partial Fraction Decomposition is:

$$\frac{Ax + B}{ax^2 + bx + c}$$

For each **irreducible** factor of the form $(ax^2 + bx + c)^n$ for $n > 1$,

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \ldots + \frac{A_n + B_n}{(ax^2 + bx + c)^n}$$

A Trivial Example

$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

Problem 4

Compute $\int \frac{x+1}{x^2+1}\,dx$

Split here

$$\frac{x+1}{x^2+1} = \frac{Ax+B}{x^2+1}$$

$A=1$

$B=1$

$$\int \boxed{\frac{x+1}{x^2+1}}\,dx = \frac{1}{2}\int \frac{2x}{\underbrace{x^2+1}_{u=x^2+1}}\,dx + \int \frac{1}{x^2+1}\,dx$$

$$= \frac{1}{2}\ln|x^2+1| + \tan^{-1}x + K$$

Example

$$\underline{2}x^2 - x + 4 = \underline{(A+B)}x^2 + Cx + 4A$$

Problem 5

Compute $\int \frac{2x^2-x+4}{x^3+4x}\,dx$ $= \int \frac{2x^2-x+4}{x(x^2+4)}\,dx$

$$\left[\frac{2x^2-x+4}{x(x^2+4)} = \underbrace{\frac{A}{x} + \frac{Bx+C}{x^2+4}}_{}\right] \times x(x^2+4)$$

$$2x^2-x+4 = A(x^2+4)+(Bx+C)x$$

$$= Ax^2+4A+Bx^2+Cx$$

$$= (A+B)x^2+Cx+4A$$

$$\begin{cases} A+B=2 \\ \boxed{C=-1} \\ 4A=4 \end{cases}$$

$$\rightarrow \boxed{A=1}$$

$$A+B=2 \rightarrow 1+B=2$$

$$\boxed{B=1}$$

$$\text{Integral} = \int\left(\frac{1}{x} \oplus \frac{x \ominus 1}{x^2+4}\right)dx = \int \frac{1}{x}\,dx + \frac{1}{2}\int\frac{2x}{x^2+4}\,dx \underbrace{- \int \frac{1}{x^2+4}\,dx}_{(I)}$$

$$(I): -\int\frac{1}{x^2+4}\,dx = -\int\frac{1}{4\left(\frac{x^2}{4}+1\right)}\,dx = -\frac{1}{4}\int\frac{1}{\frac{x^2}{2^2}+1}\,dx$$

$$= -{}^{(2)}\frac{1}{4}\int\frac{1}{\left(\underbrace{\frac{x}{2}}_{u}\right)^2+1}\left(\frac{1}{2}\right)dx = -\frac{1}{2}\tan^{-1}\frac{x}{2} + C_1$$

$u = \frac{1}{2}x$

$du = 1/2\,dx$

$\frac{d}{du}\tan^{-1}u \quad \frac{1}{u^2+1}$

$$\int\frac{1}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$\text{Integral} = \ln|x| + \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\tan^{-1}\frac{x}{2} + K$$

Example

$$b^2-4ac = (-2)^2-4(1)(3) = -2 < 0$$

Problem 6

Compute $\int \frac{7x^2-13x+13}{(x-2)(x^2-2x+3)}\,dx$

$$\left[\frac{7x^2-13x+13}{(x-2)(x^2-2x+3)} = \frac{A}{x-2} + \frac{Bx+C}{x^2-2x+3}\right](x-2)(x^2-2x+3)$$

$$7x^2-13x+13 = A(x^2-2x+3)+(Bx+C)(x-2)$$

$$= Ax^2-2Ax+3A+Bx^2-2Bx+Cx-2C$$

$$= (A+B)x^2+(-2A-2B+C)x+(3A-2C)$$

$$\begin{cases} A+B=7 \\ -2A-2B+C=-13 \\ 3A-2C=13 \end{cases}$$

$$\begin{cases} A+B \quad =7 \\ -2A-2B+C=-13 \end{cases} \longrightarrow \begin{cases} 2\cancel{A}+2\cancel{B} \quad =14 \\ -2\cancel{A}-\cancel{2}B+C=-13 \end{cases}$$

Add These Together

$$C=14+(-13)=\boxed{C=1}$$

$$3A-2C=13 \rightarrow 3A-2=13 \rightarrow 3A=15 \longrightarrow \boxed{A=5}$$

$$A+B=7 \longrightarrow 5+B=7 \longrightarrow \boxed{B=2}$$

$$\int \frac{7x^2-13x+13}{(x-2)(x^2-2x+3)}\,dx = \int\left(\frac{5}{x-2}+\frac{2x+1}{x^2-2x+3}\right)dx$$

$$= \int\frac{5}{x-2}\,dx + \int\frac{2x-2+2+1}{x^2-2x+3}\,dx$$

$$= \int\frac{5}{x-2}\,dx + \int\frac{2x-2}{x^2-2x+3}\,dx + \underbrace{\int\frac{3}{x^2-2x+3}\,dx}_{(I)}$$

$$u=x^2-2x+3$$

$$(x^2-2x)+3$$

+0 in

$$\underbrace{x^2-2x+1}-1+3$$

$$(x-1)^2+2$$

(I): $\int \frac{3}{x^2-2x+3}\,dx = \int \frac{3}{(x-1)^2+2}\,dx = \frac{3}{2}\int \frac{1}{\frac{(x-1)^2}{2}+1}\,dx$

$= \sqrt{2}\,\frac{3}{2}\int \frac{1}{\left(\frac{x-1}{\sqrt{2}}\right)^2+1}\left(\frac{1}{\sqrt{2}}\right)dx = \frac{3\sqrt{2}}{2}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right) + C_1$

$u = \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}$

$= 5\ln|x-2| + \ln|x^2-2x+3| + \frac{3\sqrt{2}}{2}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right) + C$

Example with Long Division

Partial Fraction only work for $\frac{p(x)}{q(x)}$ where deg $p <$ deg q

Problem 7

Compute $\int \frac{4x^2-3x+2}{4x^2-4x+3}\,dx$ $= \int \left(1 + \frac{x-1}{4x^2-4x+3}\right)dx$

$$\begin{array}{r} 1 \\ 4x^2-4x+3\,\overline{)\,\cancel{4x^2}-3x+2} \\ -(\cancel{4x^2}-4x+3) \\ \hline x-1 \end{array}$$

Part 1:

$$\int \frac{x-1}{4x^2-4x+3}\,dx = \int \frac{x-1}{\underbrace{(2x-1)^2+2}}\,dx$$

$\frac{d}{dx}$ is $2(2x-1)(2)$
$= 8x-4$

$$= \frac{1}{8}\int \frac{8x-8+4-4}{(2x-1)^2+2}\,dx$$

$$= \frac{1}{8}\int \frac{8x-4}{(2x-1)^2+2}\,dx - \frac{1}{8}\int \frac{4}{(2x-1)^2+2}\,dx$$

$$= \frac{1}{8}\int \frac{8x-4}{(2x-1)^2+2}\,dx - \frac{\sqrt{2}\,1}{2\,4}\int \frac{1}{\left(\frac{2x-1}{\sqrt{2}}\right)^2+1}\,\frac{2}{\sqrt{2}}\,dx$$

$$= \frac{1}{8}\ln\left|(2x-1)^2+2\right| - \frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + K$$

$$\frac{\sqrt{2}}{2}\cdot\frac{1}{4} = \frac{1}{\sqrt{2}}\cdot\frac{1}{4} = \frac{1}{4\sqrt{2}}$$

Final integral,

$$x + \frac{1}{8}\ln\left|(2x-1)^2+2\right| - \frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + K$$

$$= x + \frac{1}{8}\ln\left|4x^2-4x+3\right| - \frac{1}{4\sqrt{2}}\tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + K$$

Combining everything

Problem 8

Set up the PFD for $\frac{x^3+x^2+1}{x(x-1)(x^2+x+1)(x^2+1)^3}$

$$\frac{x^3+x^2+1}{x(x-1)(x^2+x+1)(x^2+1)^3} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2} + \frac{Ix+J}{(x^2+1)^3}$$

10 unknowns

10 equations

"10 by 10"

$A=-1$; $B=\frac{1}{8}$; $C=D=-1$; $E=\frac{15}{8}$

$F=-\frac{1}{8}$; $G=H=\frac{3}{4}$; $I=-\frac{1}{2}$; $J=\frac{1}{2}$

Repeating Quadratics Example

$$(x^2+1)^2 = x^4+2x^2+1$$

$$(Bx+C)(x^2+1) = Bx^3+Bx+Cx^2+C$$

Problem 9

Compute $\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\ dx$

$$\left[\frac{1-x+2x^2-x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}\right] x(x^2+1)^2$$

$$1-x+2x^2-x^3 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x$$

$$= \underline{Ax^4} + \underline{2Ax^2} + A + \underline{Bx^4} + \underline{Bx^2} + \underline{Cx^3} + \underline{Cx} + \underline{Dx^2} + \underline{Ex}$$

$$= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A$$

$$\begin{cases} A+B=0 \rightarrow 1+B=0 \rightarrow \boxed{B=-1} \\ \boxed{C=-1} \\ 2A+B+D=2 \rightarrow 2(1)-1+D=2 \rightarrow 1+D=2 \rightarrow \boxed{D=1} \\ C+E=-1 \rightarrow -1+E=-1 \rightarrow \boxed{E=0} \\ \boxed{A=1} \end{cases}$$

$$\int \left(\frac{1}{x} \boxed{+} \frac{(-1)x \boxed{+} (-1)}{x^2+1} \boxed{+} \frac{x}{(x^2+1)^2} \right) dx$$

$$\int \frac{1}{u^2} du = -\frac{1}{u} + C$$

$$= \int \frac{1}{x} dx - \frac{1}{2} \int \frac{2x}{x^2+1} dx - \int \frac{1}{x^2+1} dx + \frac{1}{2} \int \frac{2x}{(x^2+1)^2} dx$$

$$= \ln|x| - \frac{1}{2} \ln|x^2+1| - \tan^{-1} x - \frac{1}{2} \cdot \frac{1}{(x^2+1)} + K$$

Rationalizing Integrands

$u=\sqrt{x+4}$

Problem 10

Compute $\int \frac{\sqrt{x+4}}{x}\,dx$; Let $u^2=x+4 \rightarrow$ $x=u^2-4$, $dx=2u\,du$

$$\int \frac{\sqrt{x+4}}{x}dx = \int \frac{u}{u^2-4}(2u)du = 2\int \frac{u^2}{u^2-4}du$$

$$\begin{array}{r} 1 \\ u^2-4\overline{)\,u^2} \\ -(u^2-4) \\ \hline 4 \end{array}$$

$$= 2\int \left(1+\frac{4}{(u-2)(u+2)}\right)du$$

$$\frac{4}{(u-2)(u+2)} = \frac{A}{u-2}+\frac{B}{(u+2)}$$

$$4=A(u+2)+B(u-2) \rightarrow 4=Au+2A+Bu-2B$$

$$4 = (A+B)u + (2A - 2B)$$

$$\begin{cases} A+B=0 \rightarrow A=-B \longrightarrow \boxed{B=-1} \\ 2A-2B=4 \rightarrow 2A+2A=4 \rightarrow \boxed{A=1} \end{cases}$$

$$\int \left(1+\frac{4}{u^2-4}\right)du = 2\int\left(1+\frac{1}{u-2}-\frac{1}{u+2}\right)du$$

$$= 2u + 2\ln|u-2| - 2\ln|u+2| + C_0$$

$$= 2\left(\sqrt{x+4}\right) + 2\ln\left|\sqrt{x+4}-2\right| - 2\ln\left|\sqrt{x+4}+2\right| + K$$

Making Substitutions

Problem 11

Compute $\int \frac{\cos x}{\sin^2 x+\sin x}\,dx$; $u=\sin x$; $du = \cos x\,dx$

$$\int \frac{1}{u^2+u}\,du = \int \frac{1}{u(u+1)}\,du = \int\left(\frac{1}{u}-\frac{1}{u+1}\right)du$$

$$\frac{1}{u(u+1)} = \frac{A}{u}+\frac{B}{u+1}$$

$$1 = A(u+1)+Bu$$

$$= Au+A+Bu$$

$$\begin{cases} A+B=0 \to \boxed{B=-1} \\ \boxed{A=1} \end{cases}$$

$$= \ln|u| - \ln|u+1| + C_0$$

$$= \ln|\sin x| - \ln|\sin x+1| + K$$

Another Example

Problem 12

Compute $\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx$ Let $u^6 = x \to \begin{array}{l} x = u^{1/6} \\ dx = 6u^5\,du \end{array}$

$$\int \frac{1}{u^3-u^2}\cdot\frac{6u^5}{1}\,du = \int \frac{6u^5}{u^3-u^2}\,du = \int\left(6u^2+6u+6+\frac{6u^2}{u^2(u-1)}\right)du$$

$$\begin{array}{r} 6u^2+6u+6 \\ u^3-u^2 \overline{)\,6u^5} \\ -(6u^5-6u^4) \\ \hline 6u^4 \\ -(6u^4-6u^3) \\ \hline 6u^3 \\ -6u^3-6u^2 \\ \hline 6u^2 \end{array}$$

$$= \int\left(6u^2+6u+6+\frac{6}{u-1}\right)du$$

$$= 2u^3+3u^2+6u+6\ln|u-1| + C_0$$

$$= 2x^{\frac{1}{2}}+3x^{\frac{1}{3}}+6x^{\frac{1}{6}}+6\ln|x^{\frac{1}{6}}-1| + K$$

Weierstrass Substitution (Non-Examinable)

$$\frac{1}{\sin x + \cos x}$$

Theorem 13 (Weierstrass)

Any rational function of $\sin x$ *and* $\cos x$ *can be converted to a rational function of* t *by making the substitution* $\boxed{t = \tan \frac{x}{2}.}$

Why? One can check that $\cos \frac{x}{2} = \frac{1}{\sqrt{1+t^2}}$ and $\sin \frac{x}{2} = \frac{t}{\sqrt{1+t^2}}$. So,

$$\underbrace{\cos x = \frac{1-t^2}{1+t^2}};\quad \underbrace{\sin x = \frac{2t}{1+t^2}};\quad \underbrace{dx = \frac{2}{1+t^2}\, dt}$$

Example (Non-Examinable)

Problem 14

Compute $\int \frac{1}{3\sin x - 4\cos x}\ dx$

$$\begin{aligned}\int \frac{1}{3\sin x - 4\cos x}\ dx &= \int \frac{1}{3\left(\frac{2t}{1+t^2}\right) - 4\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2}{1+t^2}\ dt \\ &= \int \frac{1}{(t+2)(2t-1)}\ dt \\ &= \int \left(\frac{-1/5}{t+2} + \frac{2/5}{2t-1}\right) dt\end{aligned}$$

(cont.)

$$\begin{aligned}\int \frac{1}{3\sin x - 4\cos x}\,dx &= -\frac{1}{5}\ln|t+2| + \frac{2}{5}\ln|2t-1| + C_0\\ &= \frac{1}{5}\ln\left|\frac{2t-1}{t+2}\right| + C_0\\ &= \frac{1}{5}\ln\left|\frac{2\tan\frac{x}{2}-1}{\tan\frac{x}{2}+2}\right| + C\end{aligned}$$