MA 16200: Plane Analytic Geometry and Calculus II

Lecture 15: Sequences and Series Intro

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Sections Covered: 10.1

9) $$\int_3^\infty \frac{8\cos\left(\frac{\pi}{x}\right)}{x^2}\,dx \qquad \text{Let } u = \frac{\pi}{x}, \quad du = -\frac{\pi}{x^2}\,dx$$

$$= -\frac{8}{\pi}\int_3^\infty \cos\left(\frac{\pi}{x}\right)\frac{-\pi}{x^2}\,dx$$

$$= -\frac{8}{\pi}\sin\left(\frac{\pi}{x}\right)\Big]_3^\infty = \lim_{t\to\infty}\left[-\frac{8}{\pi}\sin\left(\frac{\pi}{x}\right)\right]_3^t$$

$$= \lim_{t\to\infty}\left[-\frac{8}{\pi}\sin\left(\frac{\pi}{t}\right) + \frac{8}{\pi}\sin\left(\frac{\pi}{3}\right)\right] = \lim_{t\to\infty}\left[-\frac{8}{\pi}\sin\frac{\pi}{t} + \frac{8}{\pi}\cdot\frac{\sqrt{3}}{2}\right]$$

$$= \frac{4}{\pi}\sqrt{3}$$

10) $\int_0^\infty 24\tan^{-1}(3w)\left[\frac{1}{1+9w^2}\right]dw$

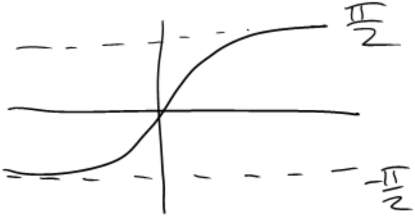
$u=\tan^{-1}3w$

$du=\frac{1}{1+(3w)^2}\cdot 3\,dx$

$$=\int_0^\infty 8\,\underbrace{\tan^{-1}(3w)}_{u}\,\underbrace{\left[\frac{3}{1+(3w)^2}\right]dx}_{du}$$

I'm going to switch to arctan(x)

$$=\lim_{t\to\infty}\left[\frac{8}{2}\arctan^2 x\right]_0^t$$



$$=\lim_{t\to\infty}\left[4\arctan^2(t)\right]$$

$$=4\left[\frac{\pi}{2}\right]^2=4\frac{\pi^2}{4}=\pi^2$$

$\int u=\frac{1}{2}u^2$

$$\int \frac{e^x}{(e^x-1)(e^x+4)}\,dx \qquad du \quad u = e^x$$

$$du = e^x\,dx$$

multiply by $(u-1)(u+4)$

$$= \int \frac{1}{(u-1)(u+4)}\,du \qquad \left[\frac{1}{(u-1)(u+4)} = \frac{A}{u-1} + \frac{B}{u+4}\right]$$

$$= \int \left(\frac{1/5}{u-1} - \frac{\frac{1}{5}}{u+4}\right) du$$

$$1 = A(u+4) + B(u-1)$$

$$= Au + 4A + Bu - B$$

$$= \frac{1}{5}\ln|u-1| - \frac{1}{5}|u+4| + C$$

$$0u + 1 = (A+B)u + (4A-B)$$

$$\begin{cases} A+B = 0 \to B = -A \to \boxed{B = -\frac{1}{5}} \\ 4A - B = 1 \to 4A + A = 1 \to \boxed{A = \frac{1}{5}} \end{cases}$$

$$= \frac{1}{5}\ln|e^x - 1| - \frac{1}{5}|e^x + 4| + C$$

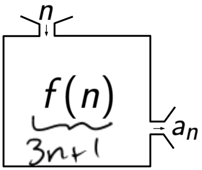
Enumerated Lists

Say we have an ordered list of numbers:

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

$$1, 4, 7, \ldots 3n+1, \ldots$$

We can think of it as a function taking non-negative integers as inputs and real numbers as outputs.



Sequence Definition

Definition 1

A **sequence** is an ordered list of numbers of the form:

$$\{a_1, a_2, \dots a_n, \dots\}$$

n-th term

The subscript n is called the **index**. The number a_n is called the ***n*-th term** in the sequence.

Notation: Usually n starts at either 0 or 1 (but it is not required)

$$\{a_n\}_{n=1}^{\infty} \qquad \{a_n\} \qquad a_n;\ n \geq 1$$

index

starting index

Defining Sequences

One can define sequences:

1. **Explicitly** using a function $a_n = f(n)$.
   - $\{\sqrt{n-3}\}_{n=3}^{\infty}$ $= \{0, 1, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n-3}, \ldots\}$
   - $\left\{\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5} \ldots, (-1)^{n+1}\frac{n}{n+1}, \ldots\right\}$
2. **Recursively** using a **recurrence relation** $a_{n+1} = f(a_n)$.
   - The **Fibonacci Sequence:**

$$f_{n+1} = f_n + f_{n-1}; \quad f_1 = 1; \quad f_0 = 1$$

1, 1

$f_2 = f_1 + f_0 = 1 + 1 = 2$

   - Given a sequence $\{a_n\}$, define the **sequence of partial sums**

$$S_N = a_N + S_{N-1}; \quad S_1 = a_1$$

$S_N = a_1 + a_2 + \cdots + a_N$

3. "Abstractly" (when there is no obvious formula)
   - Let $\{p_n\}$ be the sequence of prime numbers:

$$\{p_n\} = \{2, 3, 5, 7, 11, 13, \ldots\}$$

Example (Explicit Formulas)

$\{a_n\}_{n=1}^{\infty}$

Problem 2

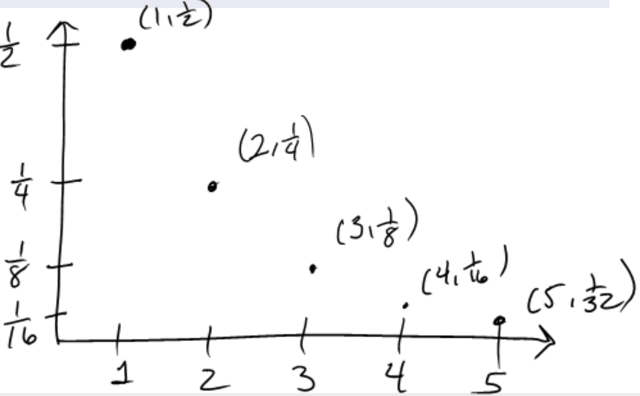
Use the explicit formula $\left\{\frac{1}{2^n}\right\}_{n=1}^{\infty}$ *to write the first 4 terms of the sequence. Sketch a graph of the sequence.*

$$a_1 = \frac{1}{2^1} = \frac{1}{2}$$

$$a_2 = \frac{1}{2^2} = \frac{1}{4}$$

$$a_3 = \frac{1}{2^3} = \frac{1}{8}$$

$$a_4 = \frac{1}{2^4} = \frac{1}{16}$$



Example (Explicit Formulas)

Alternate the signs

Problem 3

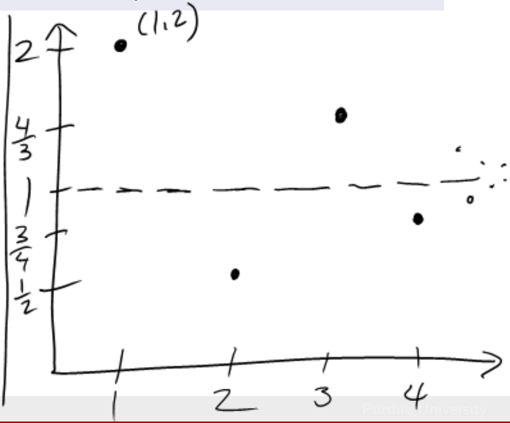
Use the explicit formula $\left\{1 + \boxed{\frac{(-1)^{n+1}}{n}}\right\}_{n=1}^{\infty}$ *to write the first 4 terms of the sequence. Sketch a graph of the sequence.*

$$a_1 = 1 + \frac{(-1)^{1+1}}{1} = 1 + 1 = 2$$

$$a_2 = 1 + \frac{(-1)^{2+1}}{2} = 1 + \frac{-1}{2} = \frac{1}{2}$$

$$a_3 = 1 + \frac{(-1)^{3+1}}{3} = 1 + \frac{1}{3} = \frac{4}{3}$$

$$a_4 = 1 + \frac{(-1)^{4+1}}{4} = 1 - \frac{1}{4} = \frac{3}{4}$$



Example (Recursion)

Problem 4

Use the recurrence relation to write the first 4 terms of the sequence.

$$\begin{cases} a_{n+1} = a_n + a_{n-1} \\ a_0 = 1 \\ a_1 = 1 \end{cases}$$

"Initial Values"
OR
"Base Case"
OR
"Seed Values"

$a_2 = a_1 + a_0 = 1 + 1 = 2$

$a_3 = a_2 + a_1 = 2 + 1 = 3$

$a_4 = a_3 + a_2 = 3 + 2 = 5$

$a_5 = a_4 + a_3 = 5 + 3 = 8$

Example (Recursion)

Problem 5

Use the recurrence relation to write the first 4 terms of the sequence.

$$\begin{cases} a_{n+1} = 2a_n + 1 \\ a_1 = 1 \end{cases}$$

$a_2 = 2 \cdot a_1 + 1 = 2 \cdot 1 + 1 = 3$

$a_3 = 2 \cdot a_2 + 1 = 2 \cdot 3 + 1 = 7$

$a_4 = 2 \cdot a_3 + 1 = 2 \cdot 7 + 1 = 15$

$a_5 = 2 \cdot a_4 + 1 = 2 \cdot 15 + 1 = 31$

$a_n = 2^n - 1$

Example (Finding Formulas)

a_n	-2	5	12	19
n	0	1	2	3

Problem 6

Consider the sequence $\{a_n\} = \{\underset{a_0}{-2}, \overset{+7}{\underset{a_1}{5}}, \underset{a_2}{12}, \boxed{\underset{a_3}{19}, \ldots}\}$.

1. *Find 2 different formulas describing this sequence.*
2. *Use either formula to find the next 2 terms in the sequence.*

① Explicit Formula: $a_n = f(n) = 7n - 2 \,;\, n \geq 0$

$7(n-1) - 2 \,;\, n \geq 1$

Recursive Formula:

$$a_{n+1} = 7(n+1) - 2 = 7n + 7 - 2 = \underbrace{(7n-2)}_{a_n} + 7$$

$$= a_n + 7$$

$$\begin{cases} a_{n+1} = a_n + 7 \\ a_0 = -2 \end{cases}$$

② $a_4 = a_3 + 7 = 19 + 7 = 26$

$a_5 = a_4 + 7 = 26 + 7 = 33$

Example (Finding Formulas)

$n!$ is the fcn s.t.
(1) Domain integers
(2) $n! = n \cdot (n-1)!$

Problem 7

Consider the sequence $\{a_n\} = \{1, -2, 6, -24, 120, \ldots\}$.

1. *Find 2 different formulas describing this sequence.*
2. *Use either formula to find the next 2 terms in the sequence.*

n	a_n
1	1
2	$-2 = (-1)\, 2 \cdot 1$
3	$6 = 3 \cdot 2 \cdot 1$
4	$-24 = (-1) \cdot 4 \cdot 3 \cdot 2 \cdot 1$
5	$120 = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

Explicit Formula

$$a_n = (-1)^{n+1} \underbrace{n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1}$$

$$= (-1)^{n+1} \cdot n!$$

Recursive:

$$a_{n+1} = (-1)^{(n+1)+1}(n+1)! = (-1)(-1)^{n+1}(n+1)\underbrace{n(n-1)(n-2)\ldots 3\cdot 2\cdot 1}_{n!}$$

$$= (-1)(n+1)\underbrace{\boxed{(-1)^{n+1}n!}}_{a_n}$$

$$= -(n+1)\cdot a_n$$

Finally,

$$\begin{cases} a_{n+1} = -(n+1)\cdot a_n \\ a_1 = 1 \end{cases}$$

Convergence

$a_n \to L$ means "a_n converges to L"

Definition 8

We say the sequence $\{a_n\}$ **converges** to a real number L (written $a_n \to L$) if

$$\lim_{n\to\infty} a_n = L$$

That is, the limit exists and equals L. Otherwise, we say the limit **diverges**.

Here is the formal definition, **you will not need to know this.**

Definition 9 (Formal Definition)

We say $a_n \to L$ if for any $\varepsilon > 0$ there is a positive integer N such that:

$$\text{If } n \geq N, \text{ then } |a_n - L| < \varepsilon$$

Example (Limits)

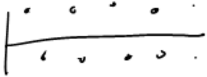
Problem 10

Make a conjecture about whether the following sequences converge or diverge. Explain why or why not.

- ① $a_n = (-1)^n \frac{3}{n+5}$; $n \geq 0$ $a_n \to 0$
- ② $a_n = \cos \pi n$; $n \geq 0$ a_n diverges (no limit point)
- ③ $a_n = 2a_n$; $a_1 = 1$ a_n is unbounded, so $\lim a_n = \infty$. So a_n diverges

① $\{a_n\} = \left\{\frac{3}{5}, -\frac{3}{6}, \frac{3}{7}, -\frac{3}{8}\right\}$

② $\{\cos \pi n\}_{n=0}^{\infty} = \{1, -1, 1, -1, \dots\}$



③ $\{a_n\} = \{1, 2, 4, 8, \dots\}$

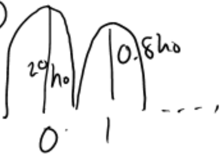
Example (Height of a Ball)

Problem 11

A basketball tossed straight up in the air reaches a high point and falls to the floor. Each time the ball bounces on the floor it rebounds to 0.8 of its previous height. Let $\{h_n\}$ *be the high point after the nth bounce, with the initial height being* $h_0 = 20ft$*.*

- *Find a recurrence relation and an explicit formula for the sequence* $\{h_n\}$*.*
- *What is the height of the peak after the 10th bounce? After the 20th bounce?*
- *Speculate the limit of the sequence* $\{h_n\}$*.*

1



$$h_0 = 20$$
$$h_1 = 0.8\,h_0$$
$$h_2 = 0.8\,h_1 = 0.8(0.8h_0) = (0.8)^2 h_0$$
$$h_3 = 0.8\,h_2 = 0.8\,(0.8)^2 h_0 = (0.8)^3 h_0$$

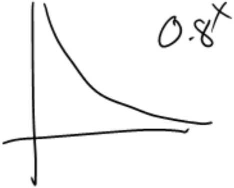
Recursive $\begin{cases} h_n = 0.8h_{n-1} \\ h_0 = 20 \end{cases}$

Explicit $h_n = (0.8)^n h_0 = 20 \cdot (0.8)^n$

2 $h_{10} = (0.8)^{10} \cdot 20 \approx 2.15\text{ ft}$

$h_{20} = (0.8)^{20} \cdot 20 \approx 0.23\text{ ft}$

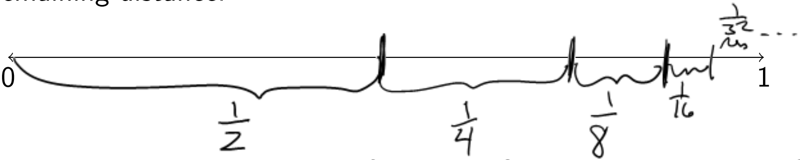
③ $\lim_{n\to\infty} h_n = \lim_{n\to\infty} 20 \cdot \underbrace{(0.8)^n}_{0<0.8<1} = 0$



$h_n \to 0$

Zeno's Paradox

Let's say we want to travel 1m. For each step, we go half the remaining distance.



What is the distance traveled after *n* steps? Will we ever reach 1m?

Distance traveled at n steps: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = \sum_{k=1}^{n} \frac{1}{2^k}$

We will at "$n = \infty$"

We want to make sense of "$\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$"

Sequence of Partial Sums

For a given sequence $\{a_n\}$, we define the **sequence of partial sums** $\{S_N\}$ as:

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$\vdots$$
$$\boxed{S_N = a_1 + a_2 + \ldots + a_N} = \underbrace{\sum_{n=1}^{N} a_n}_{\text{Explicit}} = \overbrace{a_N + S_{N-1}}^{\text{Recursive}}$$
$$\vdots$$

That is, for the infinite sum $a_1 + a_2 + \ldots + a_k + \ldots$, S_N is the value when we "chop off" the first N terms and compute its sum.

Series Definition

Definition 12

Given a sequence $\{a_n\}_{n=1}^{\infty}$, the sum of its terms:

$$a_1 + a_2 + a_3 + \ldots = \sum_{n=1}^{\infty} a_n$$

is called an **(infinite) series**. If the sequence of partial sums $\{S_N\}$ has a limit L, we say the series **converges** to L. We then write,

$$\sum_{n=1}^{\infty} a_n \overset{def}{=} \lim_{N\to\infty} \sum_{n=1}^{N} a_n = \lim_{N\to\infty} S_n = L$$

Otherwise, the series **diverges**.

$0.9999999999\ldots = 1?$

$0.bbbb\ldots = \frac{b}{9}$ for $0 \le b \le 9$

Problem 13

Make a conjecture about whether the sum:

$$\underset{0.9 + 0.09 + 0.009 + \ldots}{9/10 + 9/100 + 9/1000 + \cdots} = \sum_{n=1}^{\infty} 9 \cdot \left(\frac{1}{10}\right)^n$$

converges or diverges. If so, what is a plausible limit?

$S_1 = 0.9$

$S_2 = 0.9 + 0.09 = 0.99$

$S_3 = 0.99 + 0.009 = 0.999$

$S_4 = 0.999 + 0.0009 = 0.999$

$S_N = \sum_{n=1}^{N} 9\left(\frac{1}{10}\right)^n = 0.\underbrace{999\ldots9}_{N \text{ times}}$

$\lim_{N\to\infty} S_n$ should be 1

$$x = 0.555\ldots\ldots$$

$$10x = 5.55\ldots\ldots$$

$$\frac{-\quad x \qquad -0.55\ldots\ldots}{9x = 5}$$

$$x = \frac{5}{9}$$

Example

Telescoping Series

Problem 14

Make a conjecture about whether the sum:

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} \xrightarrow{\text{Partial Fractions}} \sum_{k=1}^{\infty} \left[\frac{1}{k} - \frac{1}{k+1}\right]$$

converges or diverges. If so, what is a plausible limit?

$S_1 = 1 - \frac{1}{2}$

$S_2 = 1 \underbrace{- \frac{1}{2} + \overbrace{\frac{1}{2}}^{a_2}}_{=0} - \frac{1}{3} = 1 - \frac{1}{3}$

$S_3 = 1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = 1 - \frac{1}{4}$

$= 1 - \frac{1}{3+1}$

$S_N = 1 - \frac{1}{N+1}$

$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{N \to \infty} S_N = \lim_{N \to \infty} \left(1 - \frac{1}{N+1}\right)$

$= 1$

The Harmonic Series

Theorem 15

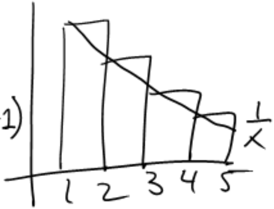
*The **harmonic series**:*

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

diverges.

Why? For full details, see Section 10.4.

$$S_N = \sum_{n=1}^{N} \frac{1}{n} > \int_{1}^{N+1} \frac{1}{x}\, dx = \ln(N+1)$$



So S_N is unbounded (hence it diverges).

Example (Distance of a Ball)

Problem 16

Suppose a ball is thrown upward to a height of h_0 meters. Each time the ball bounces, it rebounds to a fraction $\boxed{r = 0.5}$ of its previous height. Let h_n be the height after the nth bounce and let S_n be the total distance the ball has traveled at the moment of the nth bounce.

- *Find a formula for S_n and find a plausible value for the limit.*

$$S_1 = 2h_0$$

$$S_2 = S_1 + 2\left(\frac{h_0}{2}\right) = 2h_0 + \frac{1}{2}(2h_0)$$

$$= 2h_0\left(1 + \frac{1}{2}\right)$$

$$S_3 = S_2 + 2\left(\frac{h_1}{2}\right) = S_2 + 2\left(\frac{h_0}{4}\right)$$

$$S_3 = 2h_0\left(1+\frac{1}{2}+\frac{1}{4}\right)$$

$$\vdots$$

$$S_N = 2h_0\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{2^{N-1}}\right) = 2h_0\sum_{n=1}^{N}\frac{1}{2^{n-1}}$$

Find a Plausible Limit for $2h_0\underbrace{\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}}$

$$\sum_{n=1}^{\infty}\frac{1}{2^{n-1}} = \lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{2^{n-1}} = 2$$

Total Distance: $2h_0\sum_{n=1}^{\infty}\frac{1}{2^{n-1}} = 4h_0$

Comparing to Functions

	Sequences/Series	**Functions**
Independent Variable	n	x
Dependent Variable	a_n	$f(x)$
Domain	Integers	Real Numbers
Accumulation	Sums	Integrals
Accumulation over finite interval	$\sum_{n=1}^{N} a_k$	$\int_1^N f(x)\,dx$
Accumulation over an infinite interval	$\sum_{n=1}^{\infty} a_n$	$\int_1^{\infty} f(x)\,dx$