MA 16200: Plane Analytic Geometry and Calculus II

Lecture 17: Series

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Sections Covered: 10.3

(13) $a_n = \dfrac{n^9}{\ln^{27} n}$

$\ln n << x^n << e^n << n! << n^n$

is significantly large when n is large

When n is large,

$$\ln^{27} n << n^9$$

$$\lim_{n \to \infty} \frac{n^9}{\ln^{27} n} = \infty$$

Review

Recall an infinite series takes the form:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

The series converges when the sequence of partial sums $\{S_N\}$ converges. I.e., :

$$\lim_{N\to\infty} S_n \overset{def}{=} \lim_{N\to\infty} \underbrace{\sum_{n=1}^{N} a_n}_{S_n} = L$$

for some real number L. In that case, the series is equal L. Otherwise, it diverges.

Geometric Series Definition

Definition 1

A **geometric sum** is a sum of the form:

N-terms

$$S_N = \sum_{n=0}^{N-1} ar^n = \sum_{n=1}^{N} ar^{n-1} = a(\overbrace{1 + r + r^2 + r^3 + \ldots + r^{N-1}}^{N\text{-terms}})$$

$r^0 + r^1 + \cdots + r^{N-1}$

where $a \neq 0$ and r a real number. The number r is called the **common ratio**.

We eventually want to talk about the **geometric series** $\sum_{n=0}^{\infty} ar^n$

Examples

$$\sum_{n=0}^{N} ar^n = a(1+r+r^2+\cdots+r^{N-1})$$

- Examples
  - $0.99999 = \sum_{n=1}^{5} \frac{9}{10}\left(\frac{1}{10}\right)^{n-1}$ $a = 9/10$ $r = 1/10$
  - $\sum_{n=0}^{9} 3^n$ $a = 1\ ;\ r = 3$
  - $\sum_{n=1}^{3} 2\left(-\frac{3}{4}\right)^{n-1}$; $a = 2$; $r = -\frac{3}{4}$
  - $\sum_{n=0}^{\infty} 2^{-2n}5^{n+1}$ $= \sum_{n=0}^{\infty} 5\,(2^{-2})^n 5^n = \sum_{n=0}^{\infty} 5\left(\frac{1}{4}\right)^n 5^n$ $= \sum_{n=0}^{\infty} \underbrace{5}_{a=5}\underbrace{\left(\frac{5}{4}\right)^n}_{r=\frac{5}{4}}$

    $5^n \cdot 5$
- Non-Examples
  - $\sum_{k=1}^{\infty} \frac{3}{k^2+5k+4}$
  - $\sum_{i=1}^{12} i$
  - $\sum_{n=1}^{\infty} \frac{x^n}{n!}$

Partial Sum Formula

For $S_N = \sum_{n=0}^{N-1} ar^n$, can we find an explicit formula for S_N?

$$S_N = a(1 + r + r^2 + \cdots + r^{N-1})$$

$$rS_N = a(r + r^2 + r^3 + \cdots + r^N)$$

$$S_N - rS_N = a(1 + r + r^2 + \cdots + r^{N-1} - r - r^2 - r^3 - \cdots - r^{N-1} - r^N)$$

$$S_N - rS_N = a(1 - r^N)$$

$$(1-r)S_N = a(1-r^N)$$

Therefore,

$$\boxed{S_N = a\frac{1-r^N}{1-r}}$$

Value of a Geometric Series

When is $\sum_{n=0}^{\infty} ar^n < \infty$? $\sum_{n=0}^{\infty} ar^n = \lim_{N\to\infty} \frac{a(1-r^N)}{1-r}$

What values of r makes its $\lim_{N\to\infty} r^N$ exists?

$r \in (-1, 1]$. However, if $r = 1$.

$$\sum_{n=0}^{\infty} a \cdot (1)^n = \sum_{n=0}^{\infty} a \leftarrow \text{diverge}$$

When $|r| < 1$ $(-1 < r < 1)$, $r^N \to 0$ as $N \to \infty$

$$\sum_{n=0}^{\infty} ar^n = \lim_{N\to\infty} a\frac{(1-r^N)}{1-r} = \boxed{\frac{a}{1-r}}$$

Geometric Series Formula

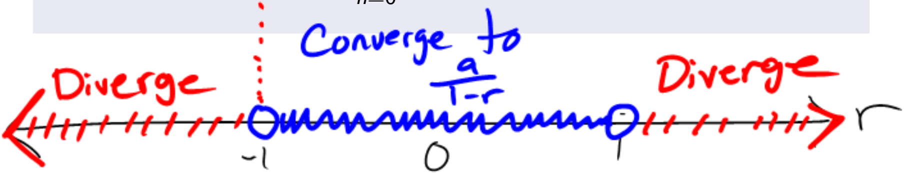
Theorem 2 (Convergence of a Geometric Series)

Let $a \neq 0$ and r be real numbers.
If $|r| < 1$, then

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

If $|r| \geq 1$,

$$\sum_{n=0}^{\infty} ar^n \text{ diverges}$$



Example

Problem 3

Compute $\sum_{n=0}^{\infty} 5\left(-\frac{2}{3}\right)^n$*, or show that it diverges.*

$a=5$ $\quad$ $r=-\frac{2}{3}$

Check if $\left|-\frac{2}{3}\right| = \frac{2}{3} < 1$ ✓. The series converges

$$\sum_{n=0}^{\infty} 5\left(-\frac{2}{3}\right)^n = \frac{a}{1-r} = \frac{5}{1-\left(-\frac{2}{3}\right)} = \frac{5}{\frac{5}{3}} = \boxed{3}$$

Example

Problem 4

Compute $\sum_{n=1}^{\infty} 1 \cdot \left(\frac{\pi}{e}\right)^{n-1}$, *or show that it diverges.*

$a = 1$ $\quad \frac{\pi}{e} = r$

We know that $e < 3 < \pi$, i.e., $e < \pi$

Thus, $\frac{\pi}{e} > 1$. $\quad \left|\frac{\pi}{e}\right| = \frac{\pi}{e} > 1$

$|r| \geq 1$, so the series diverges.

Example

$$a^b a^c = a^{b+c}$$

Problem 5

Compute $\sum_{n=1}^{\infty} 2^{2n}3^{1-n}$, *or show that it diverges.*

$$\sum_{n=1}^{\infty} 2^{2n}3^{1-n} = \sum_{n=1}^{\infty} (2^2)^n \cdot 3 \cdot 3^{-n} = \sum_{n=1}^{\infty} 3 \cdot 4^n \cdot \left(\frac{1}{3}\right)^n$$

$$= \sum_{n=1}^{\infty} 3\left(\frac{4}{3}\right)^n = \sum_{n=1}^{\infty} 3 \cdot \frac{4}{3} \cdot \left(\frac{4}{3}\right)^{n-1}$$

$$= \sum_{n=1}^{\infty} \underset{a}{\underset{\uparrow}{4}} \cdot \left(\underset{r}{\underset{\uparrow}{\frac{4}{3}}}\right)^{n-1}$$

$$\left|\frac{4}{3}\right| = \frac{4}{3} \geq 1$$

Series Diverges

Repeating Decimals

Problem 6

Convert the repeating decimal $1.\overline{2} = 1.222...$ *into a fraction.*

$$1.\overline{2} = 1 + 0.2 + 0.02 + 0.002 + 0.0002 + \cdots$$

$$= 1 + \frac{2}{10} + \frac{2}{100} + \frac{2}{1000} + \frac{2}{10\,000} + \cdots$$

$$= 1 + \frac{2}{10}\left(1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots\right)$$

$$= 1 + \sum_{n=0}^{\infty} \underbrace{\frac{2}{10}}_{a} \cdot \Big(\underbrace{\frac{1}{10}}_{r}\Big)^n = 1 + \frac{2/10}{1 - 1/10} = 1 + \frac{2/10}{9/10}$$

$\left|\frac{1}{10}\right| < 1$, converges

$$= 1 + \frac{2}{9} = \frac{11}{9}$$

Repeating Decimals

Problem 7

Convert the repeating decimal $2.3\overline{17} = 2.3171717\ldots$ *into a fraction.*

$$2.3\overline{17} = 2.3 + 0.0\overline{17} = 2.3 + 0.017 + \underline{0.00017} + 0.0000017 + \cdots$$

$$= \frac{23}{10} + \frac{17}{10^3} + \frac{17}{10^5} + \frac{17}{10^7} + \cdots$$

$$= \frac{23}{10} + \frac{17}{10^3}\left(1 + \frac{1}{10^2} + \underbrace{\frac{1}{10^4}}_{\left(\frac{1}{10^2}\right)^2} + \underbrace{\frac{1}{10^6}}_{\left(\frac{1}{10^2}\right)^3} + \cdots\right)$$

$$= \frac{23}{10} + \sum_{n=0}^{\infty} \frac{17}{10^3}\Bigg(\underbrace{\frac{1}{10^2}}_{|1/100|<1}\Bigg)^n = \frac{23}{10} + \frac{17/1000}{1-1/100} = \frac{23}{10} + \frac{17}{990} = \frac{1147}{495}$$

Sneak Peek into Power Series

Problem 8

Let f be the following function of x:

$$f(x) = \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} \overset{a=1}{} \underbrace{(-x)}_{r=-x}{}^n$$

Find the domain and range of f.

Domain: The set of all x's on which the series converges

To converge, $|-x| < 1$. In other words, $|x| < 1$, or

$$\boxed{-1 < x < 1}$$

Range:

On our domain, $f(x) = \frac{1}{1+x}$ which is decreasing, Range: $\left(\frac{1}{2}, \infty\right)$

Another Example

Problem 9

Let f be the following function of x:

$$f(x) = \sum_{n=0}^{\infty} (2x-1)^n$$

Find the domain of f.

$a = 1$
$r = 2x-1$. In order to converge, $|2x-1| < 1$

$$-1 < 2x-1 < 1$$
$$0 < 2x < 2$$
$$0 < x < 1$$

Domain : $(0,1)$

Telescoping Series

Definition 10

Telescoping Series take the form:

$$\sum_{n=1}^{\infty}[f(n) - f(n+1)]$$

for some function f

Examples:

- $\sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ $\quad f(k) = \frac{1}{k}$
- $\sum_{n=0}^{\infty}\left(e^{-n} - e^{-(n+1)}\right)$

$f(n) = e^{-n}$

Partial Sums of Telescoping Series

$\lim f(n) = \lim f(n+1)$

When does $\sum_{n=1}^{\infty}[f(n) - f(n+1)]$ converge?

$S_1 = f(1) - f(2)$

$S_2 = f(1) - f(2) + f(2) - f(3) = f(1) - f(3)$

$S_3 = f(1) - f(3) + f(3) - f(4) = f(1) - f(4)$

$\{S_N\} = \{\underbrace{f(1) - f(2)}_{N=1}, \underbrace{f(1) - f(3)}_{N=2}, \underbrace{f(1) - f(4)}_{N=3}, \ldots\}$

$\boxed{S_N = f(1) - f(N+1)}$

$\lim_{N\to\infty} S_N$ exists if and only if $\lim_{N\to\infty} f(n)$ exists

Convergence of Telescoping Series

Theorem 11 (Convergence of Telescoping Series)

If $f(n) \to L$, then:

$$\sum_{n=1}^{\infty} [f(n) - f(n+1)] = f(1) - L$$

Otherwise, ← if $\lim_{n \to \infty} f(n)$ DNE

$$\sum_{n=1}^{\infty} [f(n) - f(n+1)] \textit{ diverges}$$

Example

Problem 12

Compute $\sum_{n=1}^{\infty}\left[\cos\frac{1}{n}-\cos\frac{1}{n+1}\right]$, *or show that it diverges.*

$S_1 = \cos 1 - \cos\frac{1}{2}$

$S_2 = \cos 1 - \cancel{\cos\frac{1}{2}} + \cancel{\cos\frac{1}{2}} - \cos\frac{1}{3} = \cos 1 - \cos\frac{1}{3}$

$S_3 = \cos 1 - \cancel{\cos\frac{1}{3}} + \cancel{\cos\frac{1}{3}} - \cos\frac{1}{4} = \cos 1 - \cos\frac{1}{4}$

$\vdots$

$S_N = \cos 1 - \cos\frac{1}{N+1}$

$\sum_{n=1}^{\infty}\left[\cos\frac{1}{n}-\cos\frac{1}{n+1}\right] = \lim_{N\to\infty} S_N$

$= \lim_{N\to\infty}\left[\cos 1 - \cos\frac{1}{N+1}\right] = \cos 1 - 1$

≈ -0.46

Another way: $f(n) = \cos\frac{1}{n}$

$$\sum_{n=1}^{\infty}\left[\cos\frac{1}{n} - \cos\frac{1}{n+1}\right] = f(1) - \lim_{n\to\infty} f(n)$$

$$= \cos 1 - \cos 0 = \cos 1 - 1$$

Example

a_n

$|-1-(-2)| = 1$

$f(3) - \lim f(n)$

Problem 13

Compute $\sum_{n=3}^{\infty} \frac{1}{(n-2)(n-1)}$, *or show that it diverges.*

If S_n is the sequence of partial sums,

$$S_1 = a_3$$

$$S_2 = a_3 + a_4$$

Re-indexing: $j = n-2 \iff n = j+2$.

when $n=3$, $j=1$

$n=\infty$, $j=\infty$

$$\sum_{n=3}^{\infty} \frac{1}{(n-2)(n-1)} = \sum_{j=1}^{\infty} \frac{1}{(j+2-2)(j+2-1)} = \sum_{j=1}^{\infty} \frac{1}{j(j+1)}$$

Partial Fractions $\sum_{j=1}^{\infty} \left[\frac{1}{j} - \frac{1}{j+1}\right]$ $f(j) = 1/j$ $1 - \lim_{j\to\infty} \frac{1}{j} = 1$

Example

$\lim f(n) - f(1)$

Problem 14

Compute $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right)$, *or show that it diverges.*

$\frac{n}{n}$

$$\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) = \sum_{n=1}^{\infty} \left[\ln(n+1) + (-\ln n)\right] = \sum_{n=1}^{\infty} \left[-\ln n + \ln(n+1)\right]$$

$$= \sum_{n=1}^{\infty} \left[\underbrace{-\ln n}_{f(n) = -\ln n} - \left(-\ln(n+1)\right)\right] = -\ln 1 - \lim_{N\to\infty}(-\ln n)$$

$$= \infty$$

The series diverges

Properties of Convergent Series

$\sum_{n=1}^{N}(a_n+b_n) = \sum_{n=1}^{N} a_n + \sum_{n=1}^{N} b_n$

$\sum_{n=1}^{N} ca_n = c\sum_{n=1}^{N} a_n$

Theorem 15

Let $\sum a_n$ and $\sum b_n$ both be convergent series, then

- *For any number c, $\sum ca_n = c\sum a_n$;*
- $\sum(a_n \pm b_n) = \sum a_n \pm \sum b_n$

Theorem 16

If $\sum a_n$ diverges,

- *For $c \neq 0$, $\sum ca_k$ diverges.*
- *If $\sum b_n$ converges, $\sum(a_n \pm b_n)$ diverges.*

$\sum a_n - \sum b_n$
$= \infty - \infty$

To hold, $\sum a_n$ need to diverge and $\sum b_n$ converge

Remark

If $\sum a_n$ and $\sum b_n$ both diverge, nothing can be said about $\sum(a_n \pm b_n)$.

- $\sum a_n = \sum 1;\ \sum b_n = \sum(-1);\ \sum(a_n + b_n) = 0$
- $\sum a_n = \sum 1;\ \sum b_n = \sum 1;\ \sum(a_n - b_n) = 0$
- $\sum a_n = \sum 1;\ \sum b_n = \sum 1;\ \sum(a_n + b_n)$ diverges
- $\sum a_n = \sum 1;\ \sum b_n = \sum(-1);\ \sum(a_n - b_n)$ diverges

Example

Problem 17

Compute $\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)}+\frac{1}{2^n}\right)$

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)} \overset{\text{Part. Frac}}{=\!=} \sum_{n=1}^{\infty}\left[\frac{1}{n}-\frac{1}{n+1}\right] = \frac{1}{1}-\lim_{n\to\infty}\frac{1}{n} = 1$$

$f(n)=\frac{1}{n}$

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty}\frac{1}{2}\cdot\underbrace{\left(\frac{1}{2}\right)^{n-1}}_{|\frac{1}{2}|<1} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

$$\sum_{n=1}^{\infty}\left[\frac{3}{n(n+1)}+\frac{1}{2^n}\right] = 3\sum_{n=1}^{\infty}\frac{1}{n(n+1)}+\sum_{n=1}^{\infty}\frac{1}{2^n} = 3(1)+1$$
$$=\boxed{4}$$

Tails

$a_1 + a_2 + \cdots$ ← Both converge or

$a_M + a_{M+1} + a_{M+2} + \cdots$ Both Diverge

Theorem 18

If M is a positive integer, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=M}^{\infty} a_n$ either both converge or both diverge.

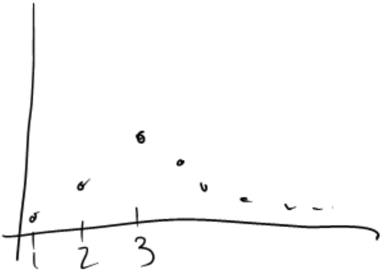
In general, when determining convergence, adding or removing finitely many terms does not change anything.

$$\sum_{n=1}^{\infty} a_n = \underbrace{\sum_{n=1}^{M} a_n}_{\text{First } M \textbf{ leading terms}} + \underbrace{\sum_{n=M+1}^{\infty} a_n}_{M\textbf{-tail}}$$

However, the *value* of the series does change if non-zero terms are added or removed.

Preview of the integral test:

- One of the conditions to use the integral test for $\sum_{n=1}^{\infty} a_n$ is a_n must be decreasing.



$$\sum_{n=1}^{\infty} a_n = \underbrace{\sum_{n=1}^{2} a_n}_{\text{finite}} + \underbrace{\sum_{n=3}^{\infty} a_n}_{\text{Test using integral test}}$$