MA 16200: Plane Analytic Geometry and Calculus II

Lecture 2: Dot and Cross Products

§13.3 The Dot Product

Definition

Definition 1 (Dot Product)

Let $\vec{u} = \langle u_1, u_2, \ldots, u_n \rangle$ and $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$ be vectors. The **dot product** of $\vec{u}$ and $\vec{v}$ is:

$$\vec{u} \cdot \vec{v} \overset{\text{def}}{=} \sum_{i=1}^{n} u_i v_i = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

In 2-Dimensions ($n = 2$):

$$\vec{u} \cdot \vec{v} = \langle u_1, u_2 \rangle \cdot \langle v_1, v_2 \rangle = u_1 v_1 + u_2 v_2$$

In 3-Dimensions ($n = 3$):

$$\vec{u} \cdot \vec{v} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Examples

Problem 2

Compute the following:

- $\langle 1, 2\rangle \cdot \langle 3, 4\rangle$
- $\langle 8, 2\rangle \cdot \langle 0, -2\rangle$
- $\langle 4, -1, 1\rangle \cdot \langle 1, 4, 0\rangle$

Sol

1. $\langle 1,2\rangle \cdot \langle 3,4\rangle = 1(3) + 2(4) = 3 + 8 = \boxed{11}$
2. $\langle 8,2\rangle \cdot \langle 0,-2\rangle = 8(0) + 2(-2) = \boxed{-4}$
3. $\langle 4,-1,1\rangle \cdot \langle 1,4,0\rangle = 4(1) + (-1)(4) + 1(0)$
   $= \boxed{0}$

Extra Space

Properties Of Dot Products

Theorem 3

Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors and c a scalar:

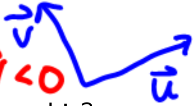
1. $\vec{0} \cdot \vec{u} = 0$
2. $\vec{u} \cdot \vec{u} = |\vec{u}|^2$
3. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ *(Commutative Property)*
4. $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$ *(Associative Property)*
5. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ *(Distributive Property)*

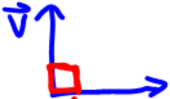
$\sqrt{\langle x,y\rangle \cdot \langle x,y\rangle} = \sqrt{x^2+y^2}$

The dot product gives a rough idea of how "aligned" two vectors are

$\vec{u} \cdot \vec{v} > 0$

$\vec{u} \cdot \vec{v} < 0$





$\vec{u} \cdot \vec{v} = 0$

. . . can we be more precise than this?

Physics Definition of The Dot Product

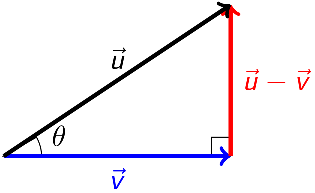
Theorem 4

*If $\vec{u}$ and $\vec{v}$ are **non-zero** vectors, then:*

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$$

where θ is the angle between $\vec{u}$ and $\vec{v}$

Why? By properties of the dot product:



$$\begin{aligned} |\vec{u} - \vec{v}|^2 &= (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) \\ &= \vec{u} \cdot \vec{u} - \vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v} \\ &= |\vec{u}|^2 - 2\vec{u} \cdot \vec{v} + |\vec{v}|^2 \\ &= |\vec{u}|^2 - 2(\vec{u} \cdot \vec{v}) + |\vec{v}|^2 \end{aligned}$$

Extra Space

$b^2 = a^2 + c^2 - 2ac\cos\theta$

Now by the Law of Cosines:

$$|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}|\ |\vec{v}|\cos\theta$$

$$|\vec{u}|^2 - 2(\vec{u}\cdot\vec{v}) + |\vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}|\ |\vec{v}|\cos\theta$$

$$-2(\vec{u}\cdot\vec{v}) = -2|\vec{u}|\ |\vec{v}|\cos\theta$$

$$\vec{u}\cdot\vec{v} = |\vec{u}|\ |\vec{v}|\cos\theta$$

Angle Between Vectors

Definition 5

1. Solving for θ we get the **angle between two vectors** $\vec{u}$ and $\vec{v}$:

$$\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right)$$

$\vec{u}\cdot\vec{v} = |\vec{u}|\,|\vec{v}|\cos\theta$

$\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}|\,|\vec{v}|}$

   If either $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$, then θ is undefined. Note that $0 \le \theta \le \pi$.

2. If $\vec{u}\cdot\vec{v} = 0$ (equivalently $\theta = \frac{\pi}{2}$), we say $\vec{u}$ and $\vec{v}$ are **orthogonal**.

In 2 and 3 dimensions, "orthogonal" and "perpendicular" mean the same thing.

Examples

$\vec{i} = \langle 1,0,0\rangle$
$\vec{j} = \langle 0,1,0\rangle$
$\vec{k} = \langle 0,0,1\rangle$

Problem 6

Use the dot product to show:

1. $2\vec{i} + 2\vec{j} - \vec{k}$ *and* $5\vec{i} - 4\vec{j} + 2\vec{k}$ *are orthogonal;*
2. $\langle 2, 1\rangle$ *is parallel to* $\langle 10, 5\rangle$.

① $\langle 2, 2, -1\rangle \cdot \langle 5, -4, 2\rangle =$

$2(5) + 2(-4) + (-1)(2) = 10 - 8 - 2 = \boxed{0}$

$\theta = \cos^{-1}\left(\frac{0}{|\vec{u}|\,|\vec{v}|}\right) = \cos^{-1}(0) = \frac{\pi}{2}$

② Show $\theta = 0$ [or π]

Extra Space

$\vec{u} = \langle 2, 1 \rangle$
$\vec{v} = \langle 10, 5 \rangle$

$\vec{u} \cdot \vec{v} = \langle 2, 1 \rangle \cdot \langle 10, 5 \rangle = 2(10) + 1(5) = 25$

$|\vec{u}| = \sqrt{2^2 + 1^2} = \sqrt{5}$

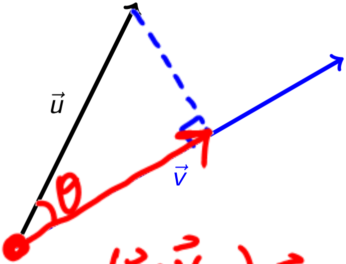
$|\vec{v}| = \sqrt{10^2 + 5^2} = \sqrt{125} = 5\sqrt{5}$

$\theta = \cos^{-1}\left(\dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{v}|}\right) = \cos^{-1}\left(\dfrac{25}{\sqrt{5}(5\sqrt{5})}\right) = \cos^{-1}(1)$

$= 0$

Projecting Onto Other Vectors

Vectors can "cast a shadow" on another vector, what is the length and direction of that "shadow"?



Length:

$$|\vec{u}|\cos\theta = \underbrace{\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}}_{\text{scal}_{\vec{v}}\vec{u}}$$

Direction:

Unit Vector in the direction of $\vec{v}$: $\dfrac{\vec{v}}{|\vec{v}|}$

Vector: $\left(\dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|}\right)\dfrac{\vec{v}}{|\vec{v}|} = \boxed{\left(\dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}}$ $\text{proj}_{\vec{v}}\vec{u}$

Components and Projections

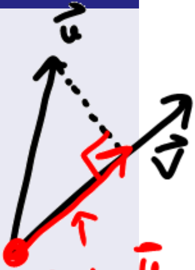
Definition 7

1. For a vector $\vec{u}$ and a non-zero vector $\vec{v}$, the **scalar component of $\vec{u}$ in the direction of $\vec{v}$ is**:

$$\mathrm{scal}_{\vec{v}}\vec{u} = |\vec{u}|\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}$$

2. The **(orthogonal) projection of $\vec{u}$ onto $\vec{v}$ is**:

$$\mathrm{proj}_{\vec{v}}\vec{u} = [\mathrm{scal}_{\vec{v}}\vec{u}]\,\frac{\vec{v}}{|v|} = \left(\frac{\vec{u}\cdot\vec{v}}{|v|^2}\right)\vec{v}$$



$\mathrm{proj}_{\vec{v}}\vec{u}$

Length: $\mathrm{scal}_{\vec{v}}\vec{u}$

Example

Problem 8

Find $\text{scal}_{\vec{v}}\vec{u}$ *and* $\text{proj}_{\vec{v}}\vec{u}$ *for* $\vec{u} = -4\vec{i} - 3\vec{j}$ *and* $\vec{v} = \vec{i} - \vec{j}$.

$$\text{scal}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|} = \frac{\langle -4,-3\rangle \cdot \langle 1,-1\rangle}{\sqrt{1^2+(-1)^2}}$$

$$= \frac{(-4)(1)+(-3)(-1)}{\sqrt{2}} = \boxed{\frac{-1}{\sqrt{2}}}$$

$$\text{proj}_{\vec{v}}\vec{u} = \left[\text{scal}_{\vec{v}}\vec{u}\right]\frac{\vec{v}}{|\vec{v}|}$$

$$= \left(\frac{-1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}\langle 1,-1\rangle = \boxed{\left\langle -\frac{1}{2}, \frac{1}{2}\right\rangle}$$

Orthogonal Decompositions

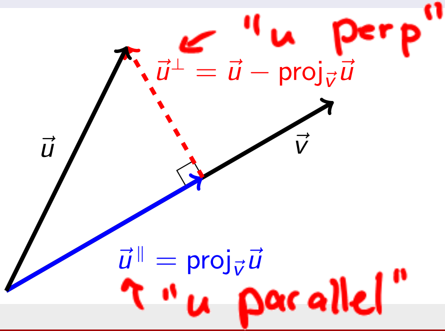
Theorem 9

For a vector $\vec{u}$ and a non-zero vector $\vec{v}$, $\vec{u}$ can be written as the sum of two vectors:

$$\vec{u} = \vec{u}^{\parallel} + \vec{u}^{\perp}$$

where $\vec{u}^{\parallel}$ is parallel to $\vec{v}$ and $\vec{u}^{\perp}$ is orthogonal to $\vec{v}$.

Why?

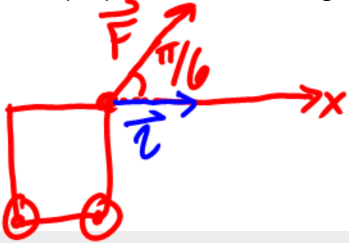


Application: Work

Problem 10

A wagon is pulled a distance of 100m along a horizontal path by a constant force of 50N. The handle of the wagon is held at an angle of 30° above the horizontal.

1) Express the force vector $\vec{F} = 50\langle \cos\frac{\pi}{6}, \sin\frac{\pi}{6}\rangle = \langle 25\sqrt{3}, 25\rangle$ as the sum of two vectors: one parallel to the ground and one perpendicular to the ground.



① $\vec{u} = \vec{F}$
$\vec{v} = \vec{i} = \langle 1, 0\rangle$

$$\vec{F}^{\parallel} = \text{proj}_{\vec{i}}\vec{F} = \frac{\vec{F}\cdot\vec{i}}{|\vec{i}|^2}\vec{i}$$
$$= (\vec{F}\cdot\vec{i})\vec{i}$$

$$\left[\langle 25\sqrt{3}, 25\rangle \cdot \langle 1,0\rangle\right] \langle 1,0\rangle = 25\sqrt{3}\langle 1,0\rangle$$

$$\vec{F}^{\parallel} = \langle 25\sqrt{3}, 0\rangle$$

$$\vec{F}^{\perp} = \vec{F} - \text{proj}_{\vec{z}}\vec{F} = \langle 25\sqrt{3}, 25\rangle - \langle 25\sqrt{3}, 0\rangle$$

$$= \langle 0, 25\rangle$$

$$\vec{F} = \underbrace{\langle 25\sqrt{3}, 0\rangle}_{\text{Parallel to } \vec{z}} + \underbrace{\langle 0, 25\rangle}_{\text{Perpendicular to } \vec{z}}$$

Work Definition

Definition 11

Given a force vector $\vec{F}$ and displacement vector $\vec{D}$, the **work (W)** done by the force is defined as:

$$W \stackrel{\text{def}}{=} (Force)(Distance) = \overbrace{(|\vec{F}|\cos\theta)}^{\text{scal}_{\vec{D}}\vec{F}}|\vec{D}| = \vec{F}\cdot\vec{D}$$

2) In the previous problem, how much work was done?

$$W = \vec{F}\cdot\vec{D} = |\vec{F}||\vec{D}|\cos\theta$$
$$= 50\cdot 100\cdot\cos\pi/6$$
$$= 50\cdot 100\cdot\frac{\sqrt{3}}{2} = 2500\sqrt{3}\ \text{Nm}$$

§13.4 The Cross Product

Review of Determinants

A **2-by-2 determinant** is defined by:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

A **3-by-3 determinant** is defined using 2-by-2 determinants:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Notice the negative sign next to the a_2.

Compute

$$\begin{vmatrix} 1 & -2 & 3 \\ 2 & 0 & 3 \\ 1 & 5 & 4 \end{vmatrix} = \underset{\substack{\uparrow \\ \text{Row 1} \\ \text{Col 1}}}{(1)} \begin{vmatrix} 0 & 3 \\ 5 & 4 \end{vmatrix} - \underset{\substack{\uparrow \\ \text{Row 1} \\ \text{Col 2}}}{(-2)} \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} + \underset{\substack{\uparrow \\ \text{Row 1} \\ \text{Col 3}}}{(3)} \begin{vmatrix} 2 & 0 \\ 1 & 5 \end{vmatrix}$$

$$= \underbrace{1\big(0(4) - (5)(3)\big)} + \underbrace{2\big(2(4) - (3)(1)\big)} + 3\big(2(5) - 0(1)\big)$$

$$= -15 + 10 + 30$$

$$= \boxed{25}$$

Cross Product Definition

Definition 12

If $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$, then the **cross product** of $\vec{u}$ and $\vec{v}$ is:

$$\vec{u} \times \vec{v} \stackrel{\text{def}}{=} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$
$$= (u_2 v_3 - u_3 v_2)\vec{i} + (u_3 v_1 - u_1 v_3)\vec{j} + (u_1 v_2 - u_2 v_1)\vec{k}$$

Notice how $\vec{u} \times \vec{v}$ is a vector and is only defined in 3-dimensions.

Examples

Problem 13

Compute $\vec{u} \times \vec{v}$ for:

1. $\vec{u} = \langle 1, 2, 0 \rangle, \vec{v} = \langle 0, 3, 1 \rangle$
2. $\vec{u} = \langle 1, 1, 0 \rangle, \vec{v} = \langle -2, 3, 0 \rangle$

① $\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 0 & 3 & 1 \end{vmatrix} = \vec{i} \begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix}$

$= (2-0)\vec{i} - (1-0)\vec{j} + (3-0)\vec{k} = 2\vec{i} - \vec{j} + 3\vec{k}$

$= \langle 2, -1, 3 \rangle$

Extra Space

② $\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 0 \\ -2 & 3 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & 0 \\ -2 & 0 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & 1 \\ -2 & 3 \end{vmatrix}\vec{k}$

$= (0-0)\vec{i} - (0-0)\vec{j} + (3-(-2))\vec{k}$

$= 5\vec{k}$

$\langle 0,0,5\rangle\cdot\vec{u} = 0$

$\langle 0,0,5\rangle\cdot\vec{v} = 0$

$(\vec{u}\times\vec{v})\cdot\vec{u} = \langle 0,0,5\rangle\cdot\langle 1,1,0\rangle = 0$

<u>Note</u> The cross product is <u>not</u> commutative

$$\vec{u} \times \vec{v} \neq \vec{v} \times \vec{u}$$

$$\vec{i} \times \vec{j} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} k$$

$$= \vec{k}$$

$$\vec{j} \times \vec{i} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \vec{k}$$

$$= (0-1) \vec{k}$$

$$= -\vec{k}$$

Properties of Cross Products

Theorem 14

Check

*$\vec{u} \times \vec{v}$ is orthogonal to **both** $\vec{u}$ and $\vec{v}$.* $(\vec{u} \times \vec{v}) \cdot \vec{u} = (\vec{u} \times \vec{v}) \cdot \vec{v} = 0$

Theorem 15

Let $\vec{u}, \vec{v}$, and $\vec{w}$ be vectors and c a scalar:

1. $\boxed{\vec{u} \times \vec{u} = \vec{0}}$ → Any parallel vectors will have a cross product of $\vec{0}$
2. $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$ *(**anti**-commutativity)*
3. $(c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v}) = c(\vec{u} \times \vec{v})$
4. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
5. $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$
6. $\vec{u} \cdot (\vec{v} \times \vec{w}) = (u \times v) \cdot \vec{w}$
7. $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$

Physics Definition of Cross Product

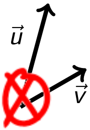
Theorem 16

*Given two **non-zero** 3-dimensional vectors $\vec{u}$ and $\vec{v}$:*

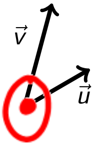
$$|\vec{u} \times \vec{v}| = |\vec{u}|\ |\vec{v}| \sin\theta$$

where $0 \leq \theta \leq \pi$ is the angle between $\vec{u}$ and $\vec{v}$.

The direction of $\vec{u} \times \vec{v}$ is given by the **right-hand rule**: Line up the tails of $\vec{u}$ and $\vec{v}$. Curl your right hand from $\vec{u}$ to $\vec{v}$, then $\vec{u} \times \vec{v}$ points in the same direction as your thumb. If $\vec{u}$ or $\vec{v}$ is $\vec{0}$, the direction is undefined.



$\otimes$ = Vector Pointing Into the page



$\odot$ = Vector Pointing out of the page

Why?

$$
\begin{aligned}
|\vec{u} \times \vec{v}|^2 =& (u_2 v_3 - u_3 v_2)^2 + (u_3 v_1 - u_1 v_3)^2 + (u_1 v_2 - u_2 v_1)^2 \\
=& u_2^2 v_3^2 - 2 u_2 u_3 v_2 v_3 + u_3^2 v_2^2 + u_3^2 v_1^2 - 2 u_1 u_3 v_1 v_3 + u_1^2 v_3^2 \\
& \quad + u_1^2 v_2^2 - 2 u_1 u_2 v_1 v_2 + u_2^2 v_1^2 \\
=& \left(u_1^2 + u_2^2 + u_3^2\right)\left(v_1^2 + v_2^2 + v_3^2\right) - (u_1 v_1 + u_2 v_2 + u_3 v_3)^2 \\
=& |\vec{u}|^2 |\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2 \\
=& |\vec{u}|^2 |\vec{v}|^2 - |\vec{u}|^2 |\vec{v}|^2 \cos^2\theta \\
=& |\vec{u}|^2 |\vec{v}|^2 \left(1 - \cos^2\theta\right) \\
=& |\vec{u}|^2 |\vec{v}|^2 \sin^2\theta
\end{aligned}
$$

Since $\sin\theta \geq 0$ for $0 \leq \theta \leq \pi$ we have $\sqrt{\sin^2\theta} = \sin\theta$, so take the square root of both sides to get $|\vec{u} \times \vec{v}| = |\vec{u}|\ |\vec{v}| \sin\theta$

Test for Parallel Vectors

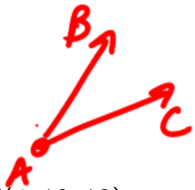
Corollary 17

1. *$\vec{u}$ and $\vec{v}$ are parallel if and only if $\vec{u} \times \vec{v} = \vec{0}$*
2. *Three points A, B, and C are collinear if and only if $\vec{AB} \times \vec{AC} = \vec{0}$*

Why? $\vec{u} \times \vec{v} = \vec{0} \Rightarrow \overbrace{0}^{|\vec{u} \times \vec{v}|} = |u|\,|v| \sin\theta$

$\Rightarrow \theta = \sin^{-1}\left(\frac{0}{|\vec{u}|\,|\vec{v}|}\right) = \sin^{-1}(0)$

$\Rightarrow \theta = 0$ or π



Example: The points $A(-2,-4,1)$, $B(1,3,7)$, and $C(4,10,13)$ are collinear.

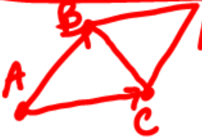
Calculating Areas and Volumes

Theorem 18

1. *The area of the parallelogram determined by* $\vec{u}$ *and* $\vec{v}$ *is* $|\vec{u} \times \vec{v}|$;
2. *The area of the triangle determined by the non-collinear points A, B, and C is* $\frac{1}{2}|\vec{AB} \times \vec{AC}|$.

Why?

Area = (Base)(Height) = $|\vec{u}|\,(|\vec{v}| \sin\theta)$
$= |\vec{u} \times \vec{v}|$



Area = $\frac{1}{2}$(Area of Parallelogram)
$= \frac{1}{2}|\vec{AB} \times \vec{AC}|$

Examples

Problem 19

1. *Determine the area of the parallelogram with adjacent sides* $\vec{u} = -4\vec{i} + 3\vec{k}$ *and* $\vec{v} = \vec{i} + \vec{j} + 2\vec{k}$.
2. *Determine the area of the triangle with vertices* $(0, 1), (-1, 1)$, *and* $(1, -1)$

① Need to compute $|\vec{u}\times\vec{v}|$

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -4 & 0 & 3 \\ 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix}\vec{i} - \begin{vmatrix} -4 & 3 \\ 1 & 2 \end{vmatrix}\vec{j} + \begin{vmatrix} -4 & 0 \\ 1 & 1 \end{vmatrix}\vec{k}$$

$$= (0-3)\vec{i} - (-8-3)\vec{j} + (-4-0)\vec{k}$$

$$= -3\vec{i} + 11\vec{j} - 4\vec{k}$$

Extra Space

$$|\vec{u}\times\vec{v}| = \sqrt{(-3)^2 + 1^2 + (-4)^2} = \sqrt{146}$$

2

$$A(0,1);\ B(1,-1);\ C(-1,1)$$

$$\vec{AB} = \langle 1,-1\rangle - \langle 0,1\rangle = \langle 1,-2\rangle$$

$$\vec{AC} = \langle -1,1\rangle - \langle 0,1\rangle = \langle -1,0\rangle$$

$$\vec{AB}\times\vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 0 \\ -1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} -2 & 0 \\ 0 & 0 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & 0 \\ -1 & 0 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & -2 \\ -1 & 0 \end{vmatrix}\vec{k}$$

$$= (0-2)\vec{k} = -2\vec{k}$$

$$\text{Area} = \frac{1}{2}(2) = 1$$

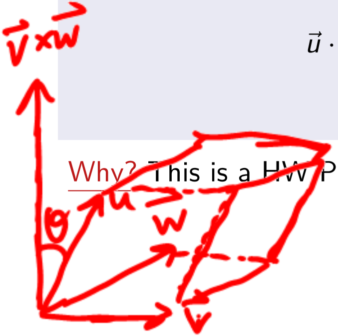
$$= (1(0) - (-1)(-2)) = -2\vec{k}$$

The Triple Scalar Product

Theorem 20

The volume of the parallelepiped determined by vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ is the absolute value of *the* ***triple scalar product****:*

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$



Why? This is a HW Problem.

$V = (\text{Base})(\text{Height})$
$= |\vec{v}\times\vec{w}|\,|\vec{u}|\cos\theta$

Example

Problem 21

Are the vectors $\vec{u} = \langle 1, 4, -7 \rangle$, $\vec{v} = \langle 2, -1, 4 \rangle$, *and* $\vec{w} = \langle 0, -9, 18 \rangle$ *coplanar?*

Check $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$

$$\text{Volume} = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix} = (-18+36) - 4(36) - 7(-18)$$

$$= 18 - 144 + 126$$

$$= 0$$

Application: Torque

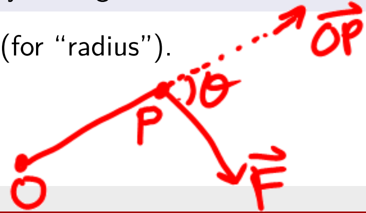
Definition 22

Applying a force $\vec{F}$ at a point P, the twisting effect (or **torque**) about a point O is a vector $\vec{\tau}$ with:

$$|\vec{\tau}| = |\overset{\vec{r}}{\vec{OP}} \times \vec{F}|$$

The direction of $\vec{\tau}$ is governed by the right hand rule.

It is common to write $\vec{OP}$ as $\vec{r}$ (for "radius").



Torque Example

$|\vec{u}\times\vec{v}| = |\vec{v}\times\vec{u}|$

Problem 23

A bolt is tightened by applying a 40-N force to a 0.25-m wrench at a 75° angle. Find the magnitude and direction of the torque about the center of the bolt.

$$|\vec{\tau}| = |\vec{OP}\times\vec{F}| = |\vec{OP}|\cdot|\vec{F}|\sin 75^\circ$$
$$= 40(0.25)\sin 75^\circ$$
$$= 10\sin 75^\circ \approx 9.66 \text{ Nm}$$

Direction : $\vec{n}$ = Unit Vector Pointing into the page

Application: Magnetism

Definition 24

A point charge q experiences a force $\vec{F}$ when it moves through a magnetic field $\vec{B}$ with velocity $\vec{v}$ governed by the equation:

$$\vec{F} = q(\vec{v} \times \vec{B})$$

Notice whether q is positive or negative influences the direction of the force.

$$|\vec{F}| = |q|\,|\vec{v}|\,|\vec{B}|\sin\theta$$

Example

Problem 25

An electron ($q = -1.6 \times 10^{-19}$ C) *enters a magnetic field* $\vec{B} = \vec{i} + \vec{j}$ *with velocity* $\vec{v} = (2 \times 10^5\ m/s)\vec{k}$.

1. *Determine the magnitude and direction of the force.*
2. *Make a rough sketch of* $\vec{v}$, $\vec{B}$, *and* $\vec{F}$.

$\vec{B}, \vec{v}$ orthogonal
$\theta = \frac{\pi}{2}$

① $|F| = |q|\,|\vec{v}|\,|\vec{B}| \sin\theta$

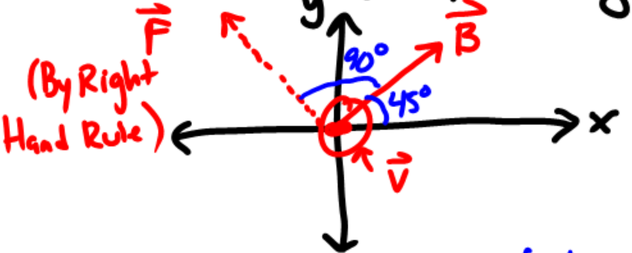
Absolute value

$= |-1.6 \times 10^{-19}|\,(2 \times 10)(\sqrt{2})(1)$

$= (3.2 \times 10^{-14})\sqrt{2}$ Newtons

Extra Space

Direction: $\vec{F}\cdot\vec{B}=0$ and $\vec{F}\cdot\vec{v}=0$, so $\vec{F}$ lies somewhere in the xy-plane



The Direction is $135°$ with respect to positive x-axis

② See Diagram Above