MA 16200: Plane Analytic Geometry and Calculus II

Lecture 20: The Alternating Series Test

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Sections Covered: 10.6

Motivation

1. We have dealt with series where the terms are always positive (integral/comparison tests). But how can we deal with this series?

$$\sum_{n=1}^{\infty} \frac{\boxed{(-1)^{n+1}}}{n}$$

Alternate the sign

2. Are there some series $\sum a_n$ where $\lim_{n\to\infty} a_n = 0$ implies the series converges?

$$\lim a_n = 0 \not\Longrightarrow \sum a_n \text{ Converges}$$

Alternating Series Test

Theorem 1 (Alternating Series Test)

The alternating series $\sum(-1)^{n+1}a_n$ *converges if:* $a_n > 0$

1. *The terms a_n are non-increasing in magnitude (eventually):*

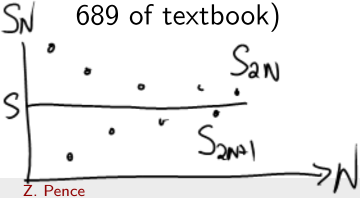
$$a_n \ge a_{n+1} > 0 \text{ for } n \text{ greater than some index } N$$

2. $\lim_{n\to\infty} a_n = 0$

Why? Use Monotone Convergence on S_{2N} and S_{2N+1} (see p.g. 689 of textbook)

Nonincreasing Bounded Seq

Nondecreasing Bounded Seq



★ The value $S = \sum (-1)^{n+1} a_n$ is sandwiched between successive partial sums

The Alternating Harmonic Series

Theorem 2

*The **alternating harmonic series** converges. Moreover,*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$$

Why? See Chapter 11

Why does it converge?

$n < n+1$

① $a_n = \frac{1}{n}$ is decreasing: $a_n = \frac{1}{n} > \frac{1}{n+1} = a_{n+1}$ when $n \geq 1$

② $\lim_{n\to\infty} a_n = 0$; $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$

Converges by the Alternating Series Test

The alternating series test is a "partial converse" for the Test for Divergence

Test for Divergence: $\boxed{\sum a_n \text{ converges}} \longrightarrow \boxed{\lim_{n\to\infty} a_n = 0}$

A.S.T.: $\boxed{\text{Alternating signs and non-increasing terms}} + \boxed{\lim_{n\to\infty} a_n = 0} \longrightarrow \boxed{\sum a_n \text{ converges}}$

$p \to q$ is "logically equivalent" to not $p \to$ not q

Example

$(-1)^n$

Problem 3

Determine whether $\sum_{n=1}^{\infty} \frac{\cos \pi n}{n^{3/4}}$ *converges or diverges. State the test used.*

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^{3/4}}$$

① $a_n = \frac{1}{n^{3/4}}$ is decreasing

Solve for n in the inequality

$$\frac{1}{(n+1)^{3/4}} < \frac{1}{n^{3/4}}$$

$$(n+1)^{3/4} > n^{3/4}$$

When $n \geq 1$,

$$\left[(n+1)^{3/4}\right]^{4/3} > \left[n^{3/4}\right]^{4/3}$$

$$n+1 > n$$

$$1 > 0 \quad \checkmark$$

② $\lim_{n\to\infty} a_n = 0$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^{3/4}} = 0$$

Converges by the Alternating Series Test

Example

Problem 4

Determine whether $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ *converges or diverges. State the test used.*

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{3n}{4n-1} = \lim_{n\to\infty} \frac{n}{n} \cdot \frac{3}{4-\frac{1}{n}} = \lim_{n\to\infty} \frac{3}{4-\frac{1}{n}}$$

$$= \frac{3}{4}$$

So $\lim_{n\to\infty} (-1)^n \frac{3n}{4n-1}$ DNE

Diverges by the Test for Divergence [n-th term Test]

Example

Problem 5

Determine whether $\sum_{n=1}^{\infty} \underbrace{(-1)^{n+1}} \frac{n^2}{n^3+1}$ *converges or diverges. State the test used.*

$\frac{n^2}{n^3+1} \to 0$

① Show $a_n = \frac{n^2}{n^3+1}$ is decreasing. Consider $f(x) = \frac{x^2}{x^3+1}$ on $x \in [1,\infty)$

$$f'(x) = \frac{2x(x^3+1) - x^2(3x^2)}{(x^3+1)^2} = \frac{x(2-x^3)}{\underbrace{(x^3+1)^2}_{\geq 0 \text{ on } (1,\infty)}}$$

Solve $x(2-x^3) \underset{\text{set}}{=} 0$ on $[1,\infty) \Rightarrow x = \sqrt[3]{2}$

$f'(2) < 0$

$\sqrt[3]{2}$ $\quad$ 2

$f(x)$ is decreasing on $(\sqrt[3]{2}, \infty)$
$f(n)$ is decreasing on $n \geq 2$

② $\lim_{n\to\infty} a_n = 0$

$$\lim_{n\to\infty} \frac{n^2}{n^3+1} = 0$$

$$(-1)^1 \frac{(1)^2}{(1)^3+1} = -\frac{1}{2}$$

So, $\underbrace{\sum_{n=1}^{\infty} a_n}_{\text{Converge}} = \underbrace{a_1}_{\text{finite}} + \underbrace{\sum_{n=2}^{\infty} \frac{(-1)^n n^2}{n^3+1}}_{\text{Converges by Alternating Series Test}}$

$\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^3+1}$ converges by the Alternating Series Test

Example

Problem 6

Determine whether $\sum_{n=2}^{\infty}(-1)^n \frac{\ln n}{n}$ converges or diverges. State the test used.

① Decreasing : $f(x) = \frac{\ln x}{x}$; $x \in [2, \infty)$

$$f'(x) = \frac{1 - \ln x}{x^2} \overset{\text{set}}{=} 0$$

$1 - \ln x = 0 \Rightarrow x = e$

$f'(3) < 0$

$e \quad 3$

$f(x)$ is decreasing on (e, ∞)

$f(n)$ is decreasing when $n \geq 3$

② $\lim_{n \to \infty} \frac{\ln n}{n} = 0$. Consider the function $f(x) = \frac{\ln x}{x}$

$$\lim_{x\to\infty} \frac{\ln x}{x} \underset{\text{L'Höpital's Rule}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

Series Converges by the Alternating Series Test

L'Höpital's Rule

f, g differentiable and $\frac{f(x)}{g(x)}$ is indeterminate at $x=a$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

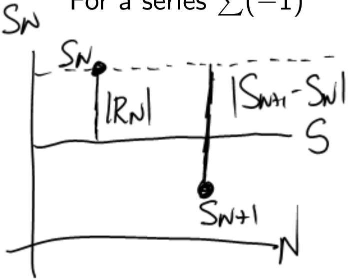
Error Term:

$$\sum_{n=1}^{\infty} a_n = \underbrace{\sum_{n=1}^{N} a_n}_{S_N} + \underbrace{\sum_{n=N+1}^{\infty} a_n}_{\text{"Error Term", } R_N}$$

Error Bound Derivation

Say $\sum (-1)^{n-1} a_n = S$

For a series $\sum(-1)^{n-1}a_n$, what is a bound for $|R_N|$?



Recall S is "sandwiched" between successive partial sums

$$|R_N| = |S - S_N| \leq |S_{N+1} - S_N|$$

$$= \left| \sum_{n=1}^{N+1} a_n - \sum_{n=1}^{N} a_n \right| = |a_{N+1}|$$

$$= a_{N+1}$$

$$|R_N| \leq a_{N+1}$$

Error Bound Formula

Theorem 7 (Remainder in Alternating Series)

Let $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ be a convergent alternating series converging to S. Let $R_N = S - S_N = \sum_{n=N+1}^{\infty}(-1)^{n+1}a_n$ be the remainder in approximating S by the sum of the first N terms. Then:

$$|R_N| \leq a_{N+1}$$

In other words, the magnitude of the remainder is less than or equal to the magnitude of the first neglected term.

Approximating ln 2

Problem 8

Recall $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = \ln 2$. *How many terms of the series are required to approximate* $\ln 2$ *with an error less than* $\varepsilon = 10^{-6}$*?*

$$|\ln 2 - S_N| = |R_N| \le a_{N+1} \overset{\text{WANT}}{<} \varepsilon$$

Want to find N where this inequality is true

$$\frac{1}{N+1} < \varepsilon$$

$$\frac{1}{\varepsilon} < N+1$$

$$N > \frac{1}{\varepsilon} - 1$$

Plug in $\varepsilon = 10^{-6}$

$$N > \frac{1}{10^{-6}} - 1$$

$$N > 10^{6} - 1$$

$$N \ge 10^{6}$$

It take a million terms to approximate $\ln(2)$ where $\varepsilon = 10^{-6}$

Approximating e^{-1}

Problem 9

Approximate $\frac{1}{e} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ *accurate to 3 decimal places.*

$$|R_N| \leq a_{N+1} \overset{\text{WANT}}{<} \frac{10^{-3}}{2} = 0.0005$$

$$\frac{1}{(n+1)!} < \frac{10^{-3}}{2}$$

$$(n+1)! > 2000$$

$$6! = 720$$

$$(6+1)! = 5040$$

$$N \geq 6$$

$$\sum_{n=0}^{6} \frac{(-1)^n}{(n+1)!} = S_7 \approx 0.36805$$

$$\frac{1}{e} \approx 0.367879$$

Approximating π

Problem 10

Leibniz's formula for π *(Proved in §11.2) states that:*

$$\pi = \sum_{n=0}^{\infty} (-1)^n \frac{4}{2n+1}$$

Bound the error for the approximation $\pi \approx \overbrace{\sum_{n=0}^{8} (-1)^n \frac{4}{2n+1}}^{S_9}$

$$|\pi - S_9| \leq a_{10} = \frac{4}{2(10)+1} = \frac{4}{21} \approx 0.1905 < \frac{10^0}{2}$$

$S_9 \approx 3.25237$

$\pi \approx 3.14159$ Only the integer part is correct

Definition

In the context of alternating series, it converges absolutely when the series still converges when we remove the oscillating part

Definition 11

1. If $\sum |a_n|$ converges, we say $\sum a_n$ is **absolutely convergent**, or $\sum |a_n|$ **converges absolutely**.
2. If $\sum |a_n|$ diverges and $\sum a_n$ converges, we say $\sum a_n$ is **conditionally convergent**, or $\sum a_n$ **converges conditionally**.

The alternating harmonic series is conditionally convergent (Why?)

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}, \text{ which diverges } \left[\begin{matrix}\text{p-series} \\ p=1\end{matrix}\right]$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{ converges by the alternating series test}$$

Example

Show $\sum |a_n|$ converges

Problem 12

Show $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ *is absolutely convergent.*

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

This is a p-series with $p=2$, so it converges absolutely.

Convergence of p-series: $p>1$

Does Absolute Convergence $\longrightarrow$ Convergence?

Yes

Abs. Conv. Implies Convergence

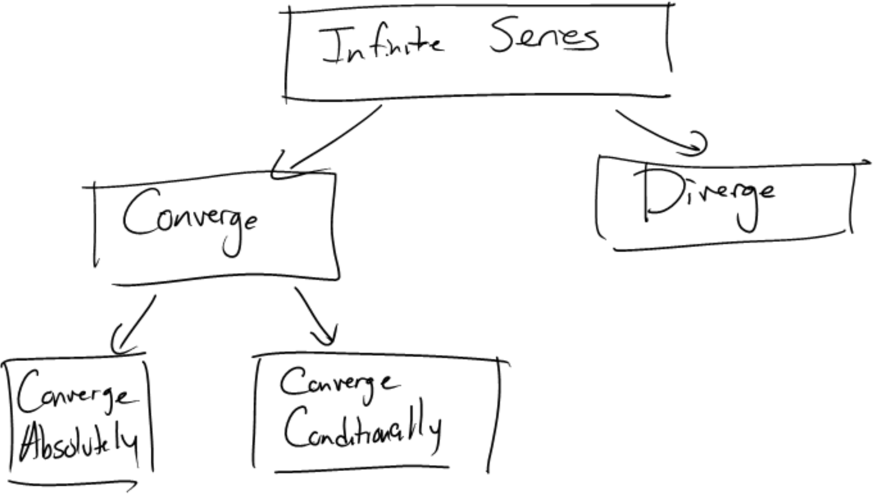
Theorem 13 (Absolute Convergence Implies Convergence)

1. *If* $\sum |a_n|$ *converges, then* $\sum a_n$ *converges.*
2. *If* $\sum a_n$ *diverges, then* $\sum |a_n|$ *diverges.*

Why? $\sum (a_n + |a_n|) \leq 2 \sum |a_n|$ converges by the comparison test. So,

$$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n| < \infty$$

(2) is the contrapositive of (1).


Infinite Series
Converge
Diverge
Converge Absolutely
Converge Conditionally

Example

Problem 14

Determine if $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ *diverges, converges absolutely, or converges conditionally.*

$$\sum_{n=1}^{\infty} \left|\frac{\sin n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{\overbrace{|\sin n|}^{\leq 1}}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$$

[Convergent p-series $(p=2)$]

$$\left|\frac{\sin n}{n^2}\right| \geq 0$$

Converges absolutely by the Comparison Test

Example

Problem 15

Determine if $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ *diverges, converges absolutely, or converges conditionally.*

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \quad \left[\text{Divergent } p\text{-series } \left(p = \tfrac{1}{2}\right) \right]$$

Not absolutely convergent by the p-series test

Consider $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

(1) Decreasing

Solve $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$

$\sqrt{n} < \sqrt{n+1}$

$0 <$ 1

(2) $\lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0$

Converges Conditionally by the Alternating Series Test

Example

Problem 16

Determine if $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n+1}$ diverges, converges absolutely, or converges conditionally.

$\frac{n}{n+1} \to 1$ as $n \to \infty$. So, $\lim_{n \to \infty} (-1)^{n+1} \frac{n}{n+1}$ DNE

Diverges by the Test for Divergence

Example

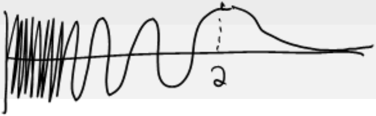
Problem 17

Determine if $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n^3}}$ *diverges, converges absolutely, or converges conditionally.*

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{\sqrt{n^3}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

Converges absolutely by the p-series test ($p = 3/2$)

Example



Problem 18

Determine if $\sum_{n=1}^{\infty}(-1)^n \sin\frac{\pi}{n}$ *converges or diverges.*

① Show $a_n = \sin\frac{\pi}{n}$ is decreasing. Consider $f(x) = \sin\frac{\pi}{x}$

$f'(x) = -\dfrac{\pi\cos(\pi/x)}{\underbrace{x^2}_{>0 \text{ on } [1,\infty)}}$ $\underline{\underline{\text{set}}}\ 0 \Rightarrow \cos\left(\frac{\pi}{x}\right) = 0$

$\frac{\pi}{x} = (2k-1)\cdot\frac{\pi}{2}$; k is a pos. integer

$x = \dfrac{2}{(2k-1)}$;

When $k=1$, $x = 2$

$f'(3) < 0$

$\rightarrow$ $f(n)$ decreasing when $n \geq 3$

② $\lim_{n \to \infty} \sin\left(\frac{\pi}{n}\right) = \sin 0 = 0$

Converges by the Alternating Series Test

Example

Problem 19

Determine if $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n\ln n}$ *converges absolutely, conditionally, or diverges.*

$$\sum_{n=1}^{\infty}\left|(-1)^n\frac{1}{n\ln n}\right| = \sum_{n=1}^{\infty}\frac{1}{n\ln n}.$$

$f(x)=\frac{1}{x\ln x}$ is decreasing, positive, and continuous on $[1,\infty)$.

But, $\int_1^{\infty}\frac{1}{x\ln x}\,dx = \lim_{a\to 1^+}\left[\ln\ln x\right]_a^2 + \lim_{b\to\infty}\left[\ln\ln x\right]_2^b$

$= \infty$

Not absolutely convergent by the integral test

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n\ln n}.$$

① a_n decreasing: $f(x)=\frac{1}{x\ln x} \Rightarrow f'(x) = -\frac{\ln x+1}{(x\ln x)^2}$ positive on $[2,\infty)$

$f'<0$ on $[2,\infty)$. ② $\lim_{n\to\infty}\frac{1}{n\ln n}=0$

Converges Conditionally by the alternating Series test.