MA 16200: Plane Analytic Geometry and Calculus II

Lecture 22: Choosing a Convergence Test

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Sections Covered: 10.8

Introduction

Keep in mind, at the end of the day, these are *strategies* for determining what test to use, not a list of rules. There are multiple ways to handle series and there are plenty of exceptions to the rules.

The main strategy: Recognize the “form” the series takes, then use the test designed to handle that form.

Test for Divergence

If it is clear that $\lim_{n\to\infty} a_n \neq 0$, then apply the Test for Divergence.

Problem 1

Determine if $\sum_{n=1}^{\infty} \left(1 + \frac{3}{n}\right)^n$ *converges or diverges.*

Geometric Series

If the series can be brought into the form $\sum_{n=1}^{\infty} ar^{n-1}$, then it is a geometric series and converges if $|r| < 1$ and diverges otherwise.

Problem 2

Compute $\sum_{n=1}^{\infty} \frac{1}{3}\left(-\frac{1}{12}\right)^{n-1}$, *or show it diverges.*

Telescoping Series

If the series can be brought into the form $\sum_{n=1}^{\infty}[a_n - a_{n+1}]$, then it is a telescoping series and converges if and only if $\lim_{n\to\infty} a_n$ exists.

Problem 3

Compute $\sum_{n=1}^{\infty}[e^{-n} - e^{-(n+1)}]$*, or show it diverges.*

p-Series

If the series can be brought into the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$, then it is a p-series and converges if $p > 1$ and diverges otherwise.

Problem 4

Determine if $\sum_{n=1}^{\infty} \frac{1}{n^{2025}}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{0.99}}$ converge or diverge.

The Comparison Tests

If the series in similar to a p-series or a geometric series, one of the comparison tests should be considered.

Problem 5

Determine if $\sum_{n=1}^{\infty} \frac{1}{2+5^n}$ *converges or diverges.*

Another Example

The Limit Comparison Test is especially useful when dealing with "algebraic functions of n" (involving roots of polynomials)

Problem 6

Determine if $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ *converges or diverges.*

The Integral Test

For $\sum a_n = \sum f(n)$, if $f(x)$ can be easily integrated, then the Integral Test is useful (assuming f satisfies the requirements).

Problem 7

Determine if $\sum_{n=1}^{\infty} ne^{-n^2}$ converges or diverges.

When the terms aren't always positive

When the terms are not always positive, it is a good idea to test for absolute convergence and use another method.

Or, use a test that tests for absolute convergence directly (such as the ratio and root tests).

The Ratio Test

Series involving factorials or other products (including a constant raised to n) are good candidates for the Ratio Test.

Problem 8

Determine if $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ *converges or diverges.*

Remark

Warning: Since for any p,

$$\left|\frac{(n+1)^p}{n^p}\right| \to 1 \text{ as } n \to \infty$$

It is best to avoid the Ratio Test if the terms are rational functions or algebraic functions of n.

The Root Test

If the terms are in the form $a_n = (b_n)^n$, then the Root Test may be useful.

Problem 9

Determine if $\sum_{n=1}^{\infty} \left(\frac{3n}{1+8n}\right)^n$ *converges or diverges.*

The Alternating Series Tests

If there is an oscillating part $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ and it is not absolutely convergent, test for conditional convergence with the Alternating Series Test.

Problem 10

Determine if $\sum_{n=1}^{\infty}(-1)^n\frac{n^3}{n^4+1}$ *converges or diverges.*

Summary (pg. 701 in textbook)

Table 10.4 Special Series and Convergence Tests

Series or Test	Form of Series	Condition for Convergence	Condition for Divergence	Comments
Geometric series	$\sum_{k=0}^{\infty} ar^k,\ a \neq 0$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	If $\lvert r\rvert < 1$, then $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$.
Divergence Test	$\sum_{k=1}^{\infty} a_k$	Does not apply	$\lim_{k\to\infty} a_k \neq 0$	Cannot be used to prove convergence
Integral Test	$\sum_{k=1}^{\infty} a_k$, where $a_k = f(k)$ and f is continuous, positive, and decreasing	$\int_1^{\infty} f(x)\,dx$ converges.	$\int_1^{\infty} f(x)\,dx$ diverges.	The value of the integral is not the value of the series.
p-series	$\sum_{k=1}^{\infty} \frac{1}{k^p}$	$p > 1$	$p \leq 1$	Useful for comparison tests
Ratio Test	$\sum_{k=1}^{\infty} a_k$	$\lim_{k\to\infty} \left\lvert\frac{a_{k+1}}{a_k}\right\rvert < 1$	$\lim_{k\to\infty} \left\lvert\frac{a_{k+1}}{a_k}\right\rvert > 1$	Inconclusive if $\lim_{k\to\infty} \left\lvert\frac{a_{k+1}}{a_k}\right\rvert = 1$
Root Test	$\sum_{k=1}^{\infty} a_k$	$\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} < 1$	$\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} > 1$	Inconclusive if $\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} = 1$
Comparison Test	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0$	$a_k \leq b_k$ and $\sum_{k=1}^{\infty} b_k$ converges.	$b_k \leq a_k$ and $\sum_{k=1}^{\infty} b_k$ diverges.	$\sum_{k=1}^{\infty} a_k$ is given; you supply $\sum_{k=1}^{\infty} b_k$.
Limit Comparison Test	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0$, $b_k > 0$	$0 \leq \lim_{k\to\infty} \frac{a_k}{b_k} < \infty$ and $\sum_{k=1}^{\infty} b_k$ converges.	$\lim_{k\to\infty} \frac{a_k}{b_k} > 0$ and $\sum_{k=1}^{\infty} b_k$ diverges.	$\sum_{k=1}^{\infty} a_k$ is given; you supply $\sum_{k=1}^{\infty} b_k$.
Alternating Series Test	$\sum_{k=1}^{\infty} (-1)^k a_k$, where $a_k > 0$	$\lim_{k\to\infty} a_k = 0$ and $0 < a_{k+1} \leq a_k$	$\lim_{k\to\infty} a_k \neq 0$	Remainder R_n satisfies $\lvert R_n\rvert \leq a_{n+1}$
Absolute Convergence	$\sum_{k=1}^{\infty} a_k$, a_k arbitrary	$\sum_{k=1}^{\infty} \lvert a_k\rvert$ converges.		Applies to arbitrary series

Splitting Series

Sometimes splitting up the series makes it easier to understand.

Problem 11

How would you approach testing the convergence of $\sum_{n=1}^{\infty} \frac{2^n + \cos(\pi n)\sqrt{n}}{3^{n+1}}$*?*

Algebra

Sometimes you need to manipulate the series to get it in a more recognizable form.

Problem 12

Determine if $\sum_{n=4}^{\infty} \frac{1}{\sqrt[4]{n^2-6n+9}}$ *converges or diverges.*

Picking a series to compare with

When using the limit comparison test, looking at the end behavior of the terms ($n \to \infty$) is useful in figuring out what series to compare with.

Problem 13

Determine if $\sum_{n=2}^{\infty} \sqrt[3]{\frac{n^2-1}{n^4+4}}$ *converges or diverges.*