MA 16200: Plane Analytic Geometry and Calculus II

Lecture 22: Choosing a Convergence Test

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Sections Covered: 10.8

Introduction

Keep in mind, at the end of the day, these are *strategies* for determining what test to use, not a list of rules. There are multiple ways to handle series and there are plenty of exceptions to the rules.

The main strategy: Recognize the “form” the series takes, then use the test designed to handle that form.

Test for Divergence

If it is clear that $\lim_{n\to\infty} a_n \neq 0$, then apply the Test for Divergence.

Problem 1

Determine if $\sum_{n=1}^{\infty}\left(1+\frac{3}{n}\right)^n$ *converges or diverges.*

$$\lim_{n\to\infty}\left(1+\frac{3}{n}\right)^n = \lim_{n\to\infty}\left[\left(1+\frac{1}{\left(\frac{n}{3}\right)}\right)^{\frac{n}{3}}\right]^3 \underset{\substack{u\to\infty \\ \text{as} \\ n\to\infty}}{\overset{u=\frac{n}{3}}{=\!=}} \lim_{u\to\infty}\left[\left(1+\frac{1}{u}\right)^u\right]^3 = e^3 \neq 0$$

Diverges by the Test for Divergence

Geometric Series

If the series can be brought into the form $\sum_{n=1}^{\infty} ar^{n-1}$, then it is a geometric series and converges if $|r| < 1$ and diverges otherwise.

Problem 2

Compute $\sum_{n=1}^{\infty} \frac{1}{3}\left(-\frac{1}{12}\right)^{n-1}$*, or show it diverges.*

Since $\left|-\frac{1}{12}\right| = \frac{1}{12} < 1$, the geometric series converges.

Moreover, $\sum_{n=1}^{\infty} \underbrace{\frac{1}{3}}_{a}\Big(\underbrace{-\frac{1}{12}}_{r}\Big)^{n-1} = \frac{\frac{1}{3}}{1-\left(-\frac{1}{12}\right)} = \frac{\frac{1}{3}}{1+\frac{1}{12}} = \frac{\frac{1}{3}}{\frac{13}{12}} = \frac{4}{13}$

Telescoping Series

If the series can be brought into the form $\sum_{n=1}^{\infty}[a_n - a_{n+1}]$, then it is a telescoping series and converges if and only if $\lim_{n\to\infty} a_n$ exists.

Problem 3

Compute $\sum_{n=1}^{\infty}[e^{-n} - e^{-(n+1)}]$, or show it diverges.

$a_n = e^{-n}$

So the N-th partial sum looks like

$$S_N = \frac{1}{e} - \frac{1}{e^{N+1}}$$

$$\sum_{n=1}^{\infty}\left[e^{-n} - e^{-(n+1)}\right] = \lim_{N\to\infty} S_N = \lim_{N\to\infty}\left(\frac{1}{e} - \frac{1}{e^{N+1}}\right) = \frac{1}{e}$$

p-Series

If the series can be brought into the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$, then it is a p-series and converges if $p > 1$ and diverges otherwise.

Problem 4

Determine if $\sum_{n=1}^{\infty} \frac{1}{n^{2025}}$ *and* $\sum_{n=1}^{\infty} \frac{1}{n^{0.99}}$ *converge or diverge.*

$$\sum_{n=1}^{\infty} \frac{1}{n^{2025}}$$

$p = 2025 > 1$

So the series Converges

$$\sum_{n=1}^{\infty} \frac{1}{n^{0.99}}$$

$p = 0.99 \leq 1$

so the series diverges

The Comparison Tests

If the series in similar to a p-series or a geometric series, one of the comparison tests should be considered.

Problem 5

Determine if $\sum_{n=1}^{\infty} \frac{1}{2+5^n}$ *converges or diverges.*

For $n \geq 1$, $0 < \frac{1}{2+5^n} \leq \frac{1}{5^n}$

Moreover, $\sum_{n=1}^{\infty} \frac{1}{5^n} = \sum_{n=1}^{\infty} \frac{1}{5}\left(\frac{1}{5}\right)^{n-1}$ is a convergent geometric series ($|\frac{1}{5}|<1$)

So $\sum_{n=1}^{\infty} \frac{1}{2+5^n}$ converges by the Comparison Test

Another Example

The Limit Comparison Test is especially useful when dealing with "algebraic functions of *n*" (involving roots of polynomials)

Problem 6

Determine if $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ *converges or diverges.*

Perform Limit Comparison with $\sum_{n=1}^{\infty} \frac{1}{3n^{3/2}}$ (which converges)

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n^3+1}}{3n^3+4n^2+2}}{\frac{1}{3n^{3/2}}} = \lim_{n\to\infty} \frac{3\sqrt{n^3}\sqrt{n^3+1}}{3n^3+4n^2+2} = \lim_{n\to\infty} 3\frac{\sqrt{n^6+n^3}}{3n^3+4n^2+2}$$

$$= \lim_{n\to\infty} 3\sqrt{\frac{n^6+n^3}{(3n^3+4n^2+2)^2}} = 3\sqrt{\lim_{n\to\infty}\frac{n^6}{n^6}\cdot\frac{1+\frac{1}{n^3}}{9+\left[\text{stuff that goes to 0 in the limit}\right]}} = 3\sqrt{\tfrac{1}{9}} = 3\cdot\tfrac{1}{3} = 1$$

When $n \geq 1$

$0<1<\infty$, So the series converges by Limit Comparison Test

Remark: In particular, for $\sum a_n$ if a_n is a rational function or algebraic function of n, $\sum a_n$ should be compared with a p-series.

The Integral Test

For $\sum a_n = \sum f(n)$, if $f(x)$ can be easily integrated, then the Integral Test is useful (assuming f satisfies the requirements).

Problem 7

Determine if $\sum_{n=1}^{\infty} ne^{-n^2}$ converges or diverges.

Let $f(x) = xe^{-x^2}$, it is Continuous and positive on $[1, \infty)$

When showing something is decreasing, when in doubt use the 1^{st} derivative test

$$f'(x) = e^{-x^2} + x(-2x)e^{-x^2} = (1-2x^2)\underbrace{e^{-x^2}}_{\text{Always} > 0} \underset{=}{\text{set}} 0$$

$1 - 2x^2 = 0 \longrightarrow x = \pm\frac{1}{\sqrt{2}} \longrightarrow f$ has no critical values on $[1, \infty)$

$f'(2) < 0$

$1/\sqrt{2}$ $\quad$ 1 $\quad\longrightarrow x$

$f(x)$ is decreasing on $\left(\frac{1}{\sqrt{2}}, \infty\right)$, so $f(n)$ is decreasing when $n \geq 1$

f meets the requirements for the Integral Test

$$\int_1^\infty x e^{-x^2}\,dx = -\frac{1}{2}\int_1^\infty e^{-x^2}(-2x)\,dx = \lim_{t\to\infty}\left[-\frac{1}{2}\int_1^t e^{\overbrace{-x^2}^{u}}\underbrace{(-2x)\,dx}_{du}\right]$$

$$= \lim_{t\to\infty} -\frac{1}{2}\left[e^{-x^2}\right]_1^t = \lim_{t\to\infty}\left[-\frac{1}{2}e^{-t^2} + \frac{1}{2}e^{-1}\right] = \frac{1}{2e}$$

Therefore, the series converges by the Integral Test

When the terms aren't always positive

When the terms are not always positive, it is a good idea to test for absolute convergence and use another method.

Or, use a test that tests for absolute convergence directly (such as the ratio and root tests).

The Ratio Test

Series involving factorials or other products (including a constant raised to n) are good candidates for the Ratio Test.

Problem 8

Determine if $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ ← a_n *converges or diverges.*

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\underbrace{\left|\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n}\right|}_{\text{Everything non-negative, drop absolute value}} = \lim_{n\to\infty}\frac{2\cdot 2^n}{2^n}\cdot\frac{n!}{(n+1)\cdot n!}$$

$$= \lim_{n\to\infty}\frac{2}{n+1} = 0 < 1$$

Series converges absolutely by the Ratio Test

Remark

Warning: Since for any p,

$$\left|\frac{(n+1)^p}{n^p}\right| \to 1 \text{ as } n \to \infty$$

It is best to avoid the Ratio Test if the terms are rational functions or algebraic functions of n.

↑ Try testing for absolute convergence by the Comparison Test instead

The Root Test

If the terms are in the form $a_n = (b_n)^n$, then the Root Test may be useful.

Problem 9

Determine if $\sum_{n=1}^{\infty} \left(\frac{3n}{1+8n}\right)^n$ *converges or diverges.*

$$\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{3n}{1+8n}\right)^n\right|} = \lim_{n\to\infty} \frac{3n}{1+8n} = \frac{3}{8} < 1$$

Series Converges absolutely by the Root Test

The Alternating Series Tests

If there is an oscillating part $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ and it is not absolutely convergent, test for conditional convergence with the Alternating Series Test.

Problem 10

Determine if $\sum_{n=1}^{\infty}(-1)^n\frac{n^3}{n^4+1}$ *converges or diverges.*

① Verify it is not absolutely convergent: Limit Compare with $\sum \frac{1}{n}$

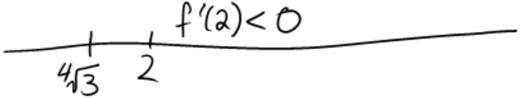
$\lim_{n\to\infty}\frac{\frac{n^3}{n^4+1}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n^4}{n^4+1} = 1$. $\sum_{n=1}^{\infty}\left|(-1)^n\frac{n^3}{n^4+1}\right|$ diverges by Limit Comparison Test

② Start A.S.T.: Show $\frac{n^3}{n^4+1}$ is non-increasing. Let $f(x)=\frac{x^3}{x^4+1}$

$$f'(x) = \frac{3x^2(x^4+1)-x^3(4x^3)}{(x^4+1)^2} \underset{=}{\text{set}} 0$$

Solving, $3x^6+3x^2-4x^6=0 \rightarrow -x^6+3x^2=0 \longrightarrow -x^2(x^4-3)=0$

On $[1,\infty)$, f has one critical at $x=\sqrt[4]{3}<2$



$f(x)$ is decreasing on $(\sqrt[4]{3},\infty)$ So $f(n)$ is decreasing eventually (when $n\geq 2$)

Also,

$$\lim_{n\to\infty}\frac{n^3}{n^4+1}=0$$

So the series converges conditionally by the Alternating Series Test

Summary (pg. 701 in textbook)

Table 10.4 Special Series and Convergence Tests

Series or Test	Form of Series	Condition for Convergence	Condition for Divergence	Comments
Geometric series	$\sum_{k=0}^{\infty} ar^k$, $a \neq 0$	$\|r\| < 1$	$\|r\| \geq 1$	If $\|r\| < 1$, then $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$.
Divergence Test	$\sum_{k=1}^{\infty} a_k$	Does not apply	$\lim_{k\to\infty} a_k \neq 0$	Cannot be used to prove convergence
Integral Test	$\sum_{k=1}^{\infty} a_k$, where $a_k = f(k)$ and f is continuous, positive, and decreasing	$\int_1^{\infty} f(x)\,dx$ converges.	$\int_1^{\infty} f(x)\,dx$ diverges.	The value of the integral is not the value of the series.
p-series	$\sum_{k=1}^{\infty} \frac{1}{k^p}$	$p > 1$	$p \leq 1$	Useful for comparison tests
Ratio Test	$\sum_{k=1}^{\infty} a_k$	$\lim_{k\to\infty} \left\|\frac{a_{k+1}}{a_k}\right\| < 1$	$\lim_{k\to\infty} \left\|\frac{a_{k+1}}{a_k}\right\| > 1$	Inconclusive if $\lim_{k\to\infty} \left\|\frac{a_{k+1}}{a_k}\right\| = 1$
Root Test	$\sum_{k=1}^{\infty} a_k$	$\lim_{k\to\infty} \sqrt[k]{\|a_k\|} < 1$	$\lim_{k\to\infty} \sqrt[k]{\|a_k\|} > 1$	Inconclusive if $\lim_{k\to\infty} \sqrt[k]{\|a_k\|} = 1$
Comparison Test	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0$	$a_k \leq b_k$ and $\sum_{k=1}^{\infty} b_k$ converges.	$b_k \leq a_k$ and $\sum_{k=1}^{\infty} b_k$ diverges.	$\sum_{k=1}^{\infty} a_k$ is given; you supply $\sum_{k=1}^{\infty} b_k$.
Limit Comparison Test	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0$, $b_k > 0$	$0 \leq \lim_{k\to\infty} \frac{a_k}{b_k} < \infty$ and $\sum_{k=1}^{\infty} b_k$ converges.	$\lim_{k\to\infty} \frac{a_k}{b_k} > 0$ and $\sum_{k=1}^{\infty} b_k$ diverges.	$\sum_{k=1}^{\infty} a_k$ is given; you supply $\sum_{k=1}^{\infty} b_k$.
Alternating Series Test	$\sum_{k=1}^{\infty} (-1)^k a_k$, where $a_k > 0$	$\lim_{k\to\infty} a_k = 0$ and $0 < a_{k+1} \leq a_k$	$\lim_{k\to\infty} a_k \neq 0$	Remainder R_n satisfies $\|R_n\| \leq a_{n+1}$
Absolute Convergence	$\sum_{k=1}^{\infty} a_k$, a_k arbitrary	$\sum_{k=1}^{\infty} \|a_k\|$ converges.		Applies to arbitrary series

Splitting Series

Sometimes splitting up the series makes it easier to understand.

Problem 11

How would you approach testing the convergence of

$\cancel{\sum_{n=1}^{\infty} \frac{2^k+\cos(\pi n)\sqrt{k}}{3^{k+1}}}$ $\sum_{n=1}^{\infty} \frac{2^n+\cos(\pi n)\sqrt{n}}{3^{n+1}}$?

$$\sum_{n=1}^{\infty} \frac{2^n+\cos(\pi n)\sqrt{n}}{3^{n+1}} = \sum_{n=1}^{\infty} \frac{2^n}{3^{n+1}} + \sum_{n=1}^{\infty} \cos(\pi n)\frac{\sqrt{n}}{3^{n+1}}$$

$$= \underbrace{\sum_{n=1}^{\infty} \frac{2}{3^2}\cdot\left(\frac{2}{3}\right)^{n-1}}_{\text{Geometric Series}} + \underbrace{\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{3^{n+1}}}_{\text{Alternating Series}}$$

Algebra

Sometimes you need to manipulate the series to get it in a more recognizable form.

Problem 12

Determine if $\sum_{n=4}^{\infty} \frac{1}{\sqrt[4]{n^2-6n+9}}$ *converges or diverges.*

$$\sum_{n=4}^{\infty} \frac{1}{\sqrt[4]{n^2-6n+9}} = \sum_{n=4}^{\infty} \frac{1}{\sqrt[4]{(n-3)^2}} = \sum_{n=4}^{\infty} \frac{1}{(n-3)^{1/2}}$$

Re-index: Let $m = n-3$

$$\sum_{n=4}^{\infty} \frac{1}{(n-3)^{1/2}} = \sum_{m=1}^{\infty} \frac{1}{m^{1/2}} \leftarrow$$ Divergent p-series ($p = \frac{1}{2}$)

Picking a series to compare with

When using the limit comparison test, looking at the end behavior of the terms ($n \to \infty$) is useful in figuring out what series to compare with.

Problem 13

Determine if $\sum_{n=2}^{\infty} \sqrt[3]{\frac{n^2-1}{n^4+4}}$ *converges or diverges.*

$\sqrt[3]{\frac{n^2-1}{n^4+4}}$ when n large $\approx$ $\sqrt[3]{\frac{n^2}{n^4}} = \sqrt[3]{\frac{1}{n^2}} = \frac{1}{n^{2/3}}$

Perform Limit Comparison with $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ [which diverges]

$\lim_{n\to\infty} \sqrt[3]{\frac{n^2-1}{n^4+4}} \cdot \sqrt[3]{n^2} = \lim_{n\to\infty} \sqrt[3]{\frac{n^4-1}{n^4+4}} = \sqrt[3]{1} = 1$

$0<1<\infty$, so the series diverges by the Limit Comparison Test