MA 16200: Plane Analytic Geometry and Calculus II

Lecture 26: Manipulaitng Power Series, Term-by-Term Differentiation/Integration

Zachariah Pence

Purdue University

Sections Covered: 11.2 (Part II)

The Radius of Convergence

Theorem 1

For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only 3 possibilities:

1. *There is a positive number R such that the series converges if $\boxed{|x-a| < R}$ and diverges if $|x-a| > R$.*
   - *This R is called the* ***Radius of Convergence****.*
2. *The series converges for all x.*
   - *By convention, $R = \infty$*
3. *The series converges only when $x = a$.*
   - *By convention, $R = 0$*

The radius of convergence is usually found by the Ratio or Root Test.

The Interval of Convergence

Definition 2

The **interval of convergence** for a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ is the set of all x where the series converges.

In the previous theorem,

1. In Case 1, the interval of convergence is 1 of 4 possibilities:

$$(a-R, a+R) \quad [a-R, a+R) \quad (a-R, a+R] \quad [a-R, a+R]$$

2. In Case 2, the interval is $\mathbb{R} = (-\infty, \infty)$.
3. In Case 3, the interval is $\{a\}$.

Geometric Series Revisited

Theorem 3

The power series centered at 0:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots = \frac{1}{1-x}$$

when $|x| < 1$.

Here the radius of convergence is 1 and the interval of convergence is $(-1, 1)$.

Example

$$\frac{1}{1-x} = \sum x^n$$

Problem 4

Find a power series representation of the function $f(x) = \frac{1}{1+2x}$. Determine the radius and interval of convergence.

Goal: Make f look like $\frac{1}{1-[\ \ ]}$

$$\frac{1}{1+2x} = \frac{1}{1-(-2x)} \overset{(*)}{=} \sum_{n=0}^{\infty} (-2x)^n = \sum_{n=0}^{\infty} (-1)^n 2^n x^n$$

$(*)$: Equality holds when $|-2x| < 1$

$$|-2| \cdot |x| < 1$$
$$2|x| < 1$$
$$|x| < \frac{1}{2}$$

$\boxed{\text{Interval: } \left(-\frac{1}{2}, \frac{1}{2}\right)}$

If $x = -\frac{1}{2}$: $\sum_{n=0}^{\infty} (-1)^n 2^n \left(-\frac{1}{2}\right)^n = \sum 1$

If $x = \frac{1}{2}$: $\sum_{n=0}^{\infty} (-1)^n 2^n \left(\frac{1}{2}\right)^n$

$$\sum_{n=0}^{\infty} (-1)^n 2^n \left(-\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} (-1)^n 2^n (-1)^n \frac{1}{2^n}$$

$$= \sum_{n=0}^{\infty} (-1)^{2n} = \sum_{n=0}^{\infty} \left[(-1)^2\right]^n = \sum_{n=0}^{\infty} 1$$

Example

Problem 5

Find a power series representation of the function $f(x) = \frac{1}{4-x}$. Determine the radius and interval of convergence.

$$\frac{1}{4-x} = \frac{1}{4}\cdot\frac{1}{1-\frac{x}{4}} \overset{(*)}{=} \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{x}{4}\right)^n = \frac{1}{4}\sum_{n=0}^{\infty}\frac{x^n}{4^n}$$

$$= \sum_{n=0}^{\infty}\frac{x^n}{4^{n+1}}$$

(*): Equality will hold when $\left|\frac{x}{4}\right| < 1$, $\frac{|x|}{4} < 1$ $\rightarrow$ $|x| < \underbrace{4}$ Radius of Convergence

Interval: $(-4, 4)$

When $x = -4$: $\sum_{n=0}^{\infty} \frac{(-4)^n}{4^{n+1}} = \sum_{n=0}^{\infty} (-1)^n \frac{4^n}{4^{n+1}} = \frac{1}{4} \sum_{n=0}^{\infty} (-1)^n$

which diverges.

When $x = 4$: $\sum_{n=0}^{\infty} \frac{4^n}{4^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{4}$, which diverges

Example

Problem 6

Find a power series representation of the function $f(x) = \frac{x^2}{1-x}$*.* *Determine the radius and interval of convergence.*

$$\frac{x^2}{1-x} = x^2\left(\frac{1}{1-x}\right) = x^2 \underbrace{\sum_{n=0}^{\infty} x^n}_{} = \sum_{n=0}^{\infty} x^{n+2}$$

Converges when this series converges. So

$|x|<1$

$x \in (-1,1)$

"is in"

Example

$$\frac{1}{1-[\quad]}$$

Problem 7

Find a power series representation of the function $f(x) = \frac{1}{1+x^2}$. Determine the radius and interval of convergence.

$$\frac{1}{1+x^2} = \frac{1}{1- -x^2} = \frac{1}{1-(-x^2)} \overset{(*)}{=} \sum_{n=0}^{\infty} (-x^2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n (x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

← General Form of an even integer

$(*)$: Converges when $|-x^2| < 1$

$|-1| \cdot |x|^2 < 1 \rightarrow |x|^2 < 1$

$|x| < 1$

Interval: $(-1, 1)$

Example

Problem 8

Determine the interval of convergence of:

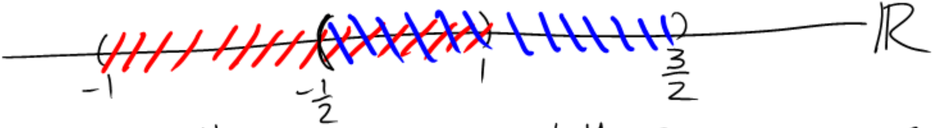
$$\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} \left(x - \frac{1}{2}\right)^n$$

What function does this power series converge to (on the interval of convergence)?

$\sum_{n=0}^{\infty} x^n$ converges on the interval $(-1,1)$

$-1 < x - \frac{1}{2} < 1$

$\sum_{n=0}^{\infty} (x-\frac{1}{2})^n$ converges when $|x-\frac{1}{2}|<1$, or $-\frac{1}{2} < x < \frac{3}{2}$



The sum will converge when both sums converge at the same time.

Interval $I = (-1,1) \cap (-\frac{1}{2}, \frac{3}{2}) = \boxed{(-\frac{1}{2}, 1)}$

Intersection ("and")

On the interval of convergence,

$$\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} \left(x-\frac{1}{2}\right)^n = \frac{1}{1-x} + \frac{1}{1-(x-\frac{1}{2})}$$

$$= \frac{1}{1-x} + \frac{1}{\frac{3}{2}-x}$$

Combining Power Series Theorem

Theorem 9

Suppose $\sum c_n x^n \to f(x)$ on an interval I_1 and $\sum d_n x^n \to g(x)$ on an interval I_2. This also applies when the center isn't 0 (it is just less obvious why).

Common Interval of Convergence

1. **Sums and Differences:** *The power series $\sum (c_n \pm d_n)x^n \to f(x) + g(x)$ on $I_1 \cap I_2$.*
2. **Multiplication by x^m:** *Suppose m is a positive integer such that $n + m \geq 0$. Then $x^m \sum c_n x^n = \sum c_n x^{m+n} \to x^m f(x)$ on I_1 (when $x \neq 0$). When $x = 0$, the series converges to $\lim_{x \to 0} x^m f(x)$.*

   $x^m \sum c_n x^n$

   $0 \cdot \infty$
3. **Composition:** *If $h(x) = bx^m$, where m is a positive integer and b a non-zero real number, then $\sum c_n (h(x))^n \to f(h(x))$ on the set of all x such that $h(x)$ is in I_1.*

   $\frac{1}{1+x^2}$

   $h(x) = -x^2$

$\sum c_n x^n \rightarrow f(x)$ when $|x| < R$

What would be the radius of convergence for

$f(ax) \leftarrow$ "$h(x) = ax$"

The series will converge when $|ax| < R$

$$|ax| < R$$

$$|a| \cdot |x| < R$$

$$|x| < \frac{R}{|a|}$$

$$\frac{1}{1 - \frac{x}{4}}$$

The new radius of convergence is $\frac{R}{|a|}$

Multiplying and Dividing Power Series (Non-Examinable)

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$:

- Define the product $f(x)g(x)$ as:

$$f(x)g(x) \stackrel{def}{=} \sum_{n=0}^{\infty} c_n x^n \text{ where } c_n = \sum_{i=0}^{n} a_i b_{n-i}$$

- If $b_0 \neq 0$, define the quotient $\frac{f(x)}{g(x)}$ as the power series $h(x) = \sum_{n=0}^{\infty} c_n x^n$ such that $f(x) = h(x)g(x)$. The coefficients c_n are found recursively:

$$\begin{cases} c_n = \frac{1}{b_0}\left[a_n - \sum_{i=1}^{n} b_i c_{n-i}\right] \\ c_0 = \frac{a_0}{b_0} \end{cases}$$

The derivative and integral of a power series

Theorem 10 (Term-by-term differentiation/integration)

Suppose a power series $\sum c_n(x-a)^n \to f(x)$ *when* $|x-a| < R$*:*

1. *Then* f *is differentiable (hence is continuous) and:*

$$f'(x) = \frac{d}{dx}\sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} c_n \frac{d}{dx}(x-a)^n = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

Equality holds on the interval of convergence

2. f *can be integrated and:*

$$\int f(x)\,dx = \int \sum_{n=0}^{\infty} c_n(x-a)^n\,dx = \sum_{n=0}^{\infty} c_n \int (x-a)^n\,dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$$

where C *is an arbitrary constant.*

3. f*,* f'*, and* $\int f\,dx$ *have the same center and* radius *of convergence.*

Why the change in index?

$$\sum_{n=0}^{\infty} C_n (x-a)^n = C_0 + C_1 (x-a) + C_2 (x-a)^2 + \cdots$$

$$\frac{d}{dx} \sum_{n=0}^{\infty} C_n (x-a)^n = 0 + C_1 + 2C_2 (x-a) + \cdots$$

$$= \sum_{n=1}^{\infty} n C_n (x-a)^{n-1}$$

The "0-th" term is just 0

Example

Problem 11

Find a power series representation of $f(x) = \frac{1}{(1-x)^2}$. Determine the radius and interval of convergence.

$$f(x) = \frac{1}{(1-x)^2}$$

$$\int f(x)\,dx = (-1)\int \frac{(-1)}{(1-x)^2}\,dx$$

$$= \frac{1}{1-x} + C$$

$$\int f(x)\,dx = C + \frac{1}{1-x} = C + \sum_{n=0}^{\infty} x^n$$

$$\frac{d}{dx}\int f(x)\,dx = \frac{d}{dx}\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \frac{d}{dx}(x^n) = \sum_{n=1}^{\infty} n x^{n-1}$$

Re-indexing: Let $m = n-1 \Leftrightarrow n = m+1$

$$f(x) = \sum_{m=0}^{\infty} (m+1) x^m$$

We know $\int f(x)\,dx = C + \sum_{n=0}^{\infty} x^{\wedge}$ has a radius of convergence of 1.

So, the radius of convergence of f at 0 is also 1.

Interval: $(-1, 1)$

When $x=-1$: $\sum_{m=0}^{\infty} (m+1)(-1)^m$ diverges

When $x=1$: $\sum_{m=0}^{\infty} (m+1)(1)^m = \sum_{m=0}^{\infty} (m+1)$ diverges

Example

Problem 12

Find a power series representation of $f(x) = \ln(1-x)$. Determine the radius and interval of convergence.

$f(x) = \ln(1-x)$

$f'(x) = \frac{-1}{1-x}$

$\underline{\underline{\text{when } |x|<1}} \; -\sum_{n=0}^{\infty} x^n$

$f'(x) = \sum_{n=0}^{\infty} x^n$

$f(x) = \int f'(x)\,dx = -\int \sum_{n=0}^{\infty} x^n \,dx$

$\underline{\underline{\text{on IOC}}} \; -\sum_{n=0}^{\infty} \int x^n\,dx = C - \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$

① Check $x=0$ is within the radius of convergence

$|x|<1$, $|0| = 0 < 1$

So ② Find C

$$\ln(1-x) = C - \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

$$0 = \ln(1-0) = C - \sum_{n=0}^{\infty} \frac{0^{n+1}}{n+1} = C + 0 = C$$

So $C=0$. Therefore,

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \overset{m=n+1}{=\!=} \boxed{-\sum_{m=1}^{\infty} \frac{x^m}{m}}$$

$f'(x) = -\sum_{n=0}^{\infty} x^n$ has a ROC of 1, so f has a ROC of 1

When $x=1$: $-\sum_{m=1}^{\infty} \frac{1^m}{m} = -\sum_{m=1}^{\infty} \frac{1}{m}$ diverges

When $x=-1$: $-\sum_{m=1}^{\infty} \frac{(-1)^m}{m}$ Converges by Alternating Series Test

Interval: $[-1, 1)$

☆ Even though f, f', and $\int f\,dx$ have the same radius of convergence, they may not have the same interval of convergence.

Example

Problem 13

Find a power series representation of $f(x) = \ln(1+x)$. Determine the radius and interval of convergence.

$$\ln(1+x) = \ln(1-(-x)) \overset{\text{When } -1 \le -x < 1}{=\!=} -\sum_{n=1}^{\infty} \frac{(-x)^n}{n}$$

$$= -\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$$

Solve $-1 \le -x < 1$, the interval of convergence

$1 \ge x > -1$, is $(-1, 1]$

Example

Problem 14

Find a power series representation of $f(x) = \ln\sqrt{1-x^2}$.
Determine the radius and interval of convergence.

$$\ln(1-x^2)^{\frac{1}{2}} = \frac{1}{2}\ln(1-x^2) = \frac{1}{2}\sum_{n=1}^{\infty}(-1)\frac{(x^2)^n}{n}$$

$$= -\frac{1}{2}\sum_{n=1}^{\infty}\frac{x^{2n}}{n}$$

When $x=\pm 1$: $-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(\pm 1)^{2n}}{n} = -\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$ diverges

Interval of Convergence: $(-1,1)$

Example

Problem 15

Find a power series representation of $f(x) = \tan^{-1} x$. Determine the radius and interval of convergence.

$f(x) = \tan^{-1} x$

$f'(x) = \dfrac{1}{1+x^2}$

$= \dfrac{1}{1-(-x^2)}$

$\underline{\underline{\text{When } |-x^2|<1}} \displaystyle\sum_{n=0}^{\infty} (-x^2)^n$

$f'(x) = \displaystyle\sum_{n=0}^{\infty} (-1)^n x^{2n}$

$f(x) = \int f'(x)\,dx = \displaystyle\int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,dx$

$\underline{\underline{\text{on IOC}}} \displaystyle\sum_{n=0}^{\infty} \int (-1)^n x^{2n}\,dx$

$f(x) = C + \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

0 is within the ROC

$$0 = \tan^{-1} 0 = C + \sum_{n=0}^{\infty} (-1)^n \frac{0^{2n+1}}{2n+1} = C + 0 = C$$

$C = 0$

So, $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

ROC is 1 since ROC of f' is 1

When $x = -1$: $\sum_{n=0}^{\infty} \frac{(-1)^n (-1)^{\overbrace{2n+1}^{\text{Odd Number}}}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)(-1)^n}{2n+1}$

Converges by Alternating Series Test

When $x = 1$: $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$

Interval of Convergence: $[-1, 1]$

Example

Problem 16

Find a power series representation of $f(x) = \ln \frac{1+x}{1-x}$. Determine the radius and interval of convergence.

$$\ln(1+x) - \ln(1-x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} + \sum_{n=1}^{\infty} \frac{x^n}{n}$$

$$= \sum_{n=1}^{\infty} \underbrace{\left[(-1)^{n+1} + 1\right]}_{\begin{cases} 2 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}} \frac{x^n}{n}$$

$$= \sum_{k=0}^{\infty} 2 \frac{x^{2k+1}}{2k+1}$$

$$\text{IOC}: (-1,1) = [-1,1) \cap (-1,1]$$

Example

$$(n+1)! = (n+1)\cdot n!$$

Problem 17

We will see in the next section that:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}; \quad x \in (-\infty, \infty)$$

ROC $= \infty$

Use this to find a power series representation for $\sin x$. *Determine the radius and interval of convergence.*

$$\sin x = \int \cos x \, dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \, dx \overset{\text{On IOC}}{=\!=} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \int x^{2n} \, dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{[(2n)!]} \cdot \frac{x^{2n+1}}{(2n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

ROC: ∞

IOC: $(-\infty, \infty)$

Application (Differential Equations)

Problem 18

Show that the series $\underbrace{f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}}_{=e^x}$ *is a solution to the Initial Value Problem (IVP):*

You may take for granted IOC is $(-\infty, \infty)$

$$\begin{cases} f'(x) = f(x) \\ f(0) = 1 \end{cases}$$

$$f'(x) = \frac{d}{dx}\sum_{n=0}^{\infty}\frac{x^n}{n!} \overset{\text{On IOC}}{=\!=} \sum_{n=0}^{\infty}\frac{1}{n!}\frac{d}{dx}(x^n) = \sum_{n=1}^{\infty}\frac{n x^{n-1}}{n!}$$

$$= \sum_{n=1}^{\infty}\frac{n x^{n-1}}{n(n-1)!} = \sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!} \overset{m=n-1}{=\!=} \sum_{m=0}^{\infty}\frac{x^m}{m!} = f(x)$$

0 is within the ROC

$$f(0) = 1 + \sum_{n=1}^{\infty} \frac{0^n}{n!} = 1$$

Ultimately $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{m=0}^{\infty} \frac{x^m}{m!}$