MA 16200: Plane Analytic Geometry and Calculus II

Lecture 29: Introduction to Polar Coordinates

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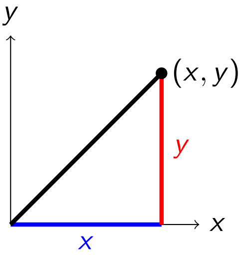
Sections Covered: 12.2 (Part I)

Cartesian Coordinates

Our usual system is called **Cartesian coordinates** (or Rectangular coordinates or Box coordinates). A point in space is described using the ordered pair

$$(x, y)$$

where x is the horizontal component and y is the vertical component.

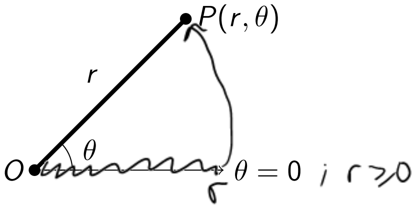


Polar Coordinates (When $r \geq 0$)

In **Polar Coordinates**, a point in space is described using the ordered pair

$$(r, \theta)$$

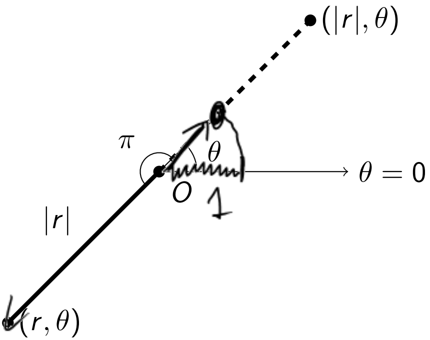
where r (called the **radial coordinate**) is the distance from the origin (called the **pole**) and θ is angle the ray $\overrightarrow{OP}$ makes with the positive x-axis (called the **polar axis**). Positive angles are measured counterclockwise.



Polar Coordinates (When $r < 0$)

When r is negative,

$$(r, \theta) \overset{def}{=} (|r|, \theta + \pi)$$



Example

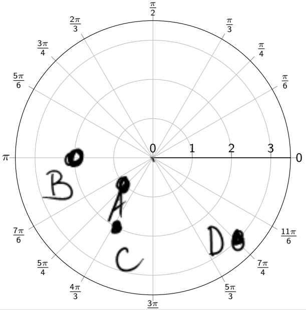
Problem 1

Plot the points whose polar coordinates are given:

$$(a)\ (1, 5\pi/4) \quad (b)\ (2, 3\pi) \quad (c)\ (2, -2\pi/3) \quad (d)\ (-3, 3\pi/4)$$

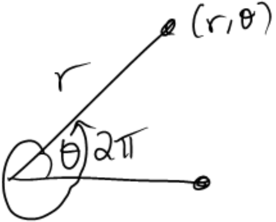
$$\left(3, \frac{3\pi}{4} + \pi\right)$$

$$\left(3, \frac{7\pi}{4}\right)$$



Representations are not Unique

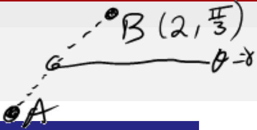
Unlike in Cartesian coordinates, there are multiple ways to describe the same point in space. Why?



$$(r,\theta) = (r, \theta + 2\pi) = (r, \theta + 2\pi n)$$ n an integer

$$(r,\theta) = (-r, \theta + \pi)$$

In practice, we restrict $\boxed{r \geq 0 \text{ and } \theta \in (-\pi, \pi]}$ to make it unique (although this is not required). This is useful in plotting points, not necessarily when plotting curves.



Example

Problem 2

Give two alternative representations for each point:

$$(a)\ (1, 5\pi/4) \quad (b)\ (2, 3\pi) \quad (c)\ (2, -2\pi/3) \quad (d)\ (-3, 3\pi/4)$$

(a) $(1, \frac{5\pi}{4}) = (1, \frac{5\pi}{4} + 2\pi) = \boxed{(1, \frac{13}{4}\pi)}$

$= (1, \frac{5\pi}{4} - 2\pi) = \boxed{(1, -\frac{3\pi}{4})}$

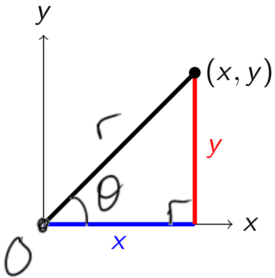
(b) $(2, 3\pi) = \boxed{(2, 5\pi)} = \boxed{(2, \pi)}$

(c) $(2, -\frac{2\pi}{3} - 2\pi) = \boxed{(2, -\frac{8\pi}{3})}$; $(2, -\frac{2\pi}{3}) = (-2, \frac{-2\pi}{3} + \pi) = \boxed{(-2, \frac{\pi}{3})}$

(d) $(-3, \frac{3\pi}{4}) = \boxed{(3, \frac{7\pi}{4})} = (3, \frac{7\pi}{4} - 2\pi) = \boxed{(3, -\frac{\pi}{4})}$

From Cartesian to Polar

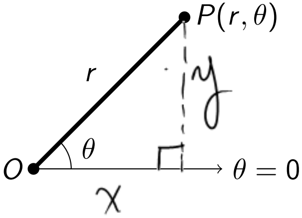
How can we convert a point in Cartesian coordinates to Polar?



$$\begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \dfrac{y}{x} \end{cases}$$

From Polar to Cartesian

How can we convert a point in Polar coordinates to Cartesian?



$$\frac{x}{r} = \cos\theta$$

$$\frac{y}{r} = \sin\theta$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Summary of Formulas

Theorem 3 (Converting Coordinates)

A point with polar coordinates (r, θ) *has Cartesian coordinates* (x, y)*, where:*

$$x = r\cos\theta \quad y = r\sin\theta$$

A point with Cartesian coordinates (x, y) *has polar coordinates* (r, θ)*, where:*

$$r^2 = x^2 + y^2 \quad \tan\theta = \frac{y}{x}$$

Example

Problem 4

Express the point with polar coordinates $P(2, \frac{3\pi}{4})$ in Cartesian coordinates.

$$x = r\cos\theta = 2\cos\frac{3\pi}{4} = 2\left(-\frac{1}{\sqrt{2}}\right)$$

$$= -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

$$y = r\sin\theta = 2\sin\frac{3\pi}{4} = 2\left(\frac{1}{\sqrt{2}}\right)$$

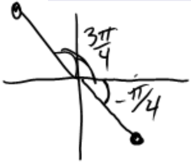
$$= \sqrt{2}$$

$$(x, y) = (-\sqrt{2}, \sqrt{2})$$

Example

Problem 5

Express the point with Cartesian coordinates $Q(1,-1)$ in Polar Coordinates.



$$r^2 = x^2 + y^2 = 1^2 + (-1)^2 = 2$$

$$\tan\theta = \frac{y}{x} = -1$$

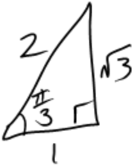
Personally, I prefer if $r \geq 0$ and $\theta \in (-\pi, \pi]$

$$(r, \theta) = \left(\sqrt{2}, -\frac{\pi}{4}\right)$$

Example

Problem 6

Express the point with polar coordinates $P(2, \frac{\pi}{3})$ in Cartesian coordinates.



$$x = r\cos\theta = 2\cos\frac{\pi}{3} = 2\left(\frac{1}{2}\right) = 1$$

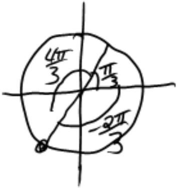
$$y = r\sin\theta = 2\sin\frac{\pi}{3} = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

$$(x, y) = (1, \sqrt{3})$$

Example

Problem 7

Express the point with Cartesian coordinates $(-4, -4\sqrt{3})$ *in Polar Coordinates.*



$$r^2 = x^2 + y^2 = (-4)^2 + (-4\sqrt{3})^2$$
$$= 16 + 48 = 54$$

$$\tan\theta = \frac{y}{x} = \frac{-4\sqrt{3}}{-4} = \sqrt{3}$$

OR

Keeping $\theta \in (-\pi, \pi]$

$$(r, \theta) = \left(\sqrt{54}, -\frac{2\pi}{3}\right) = \left(3\sqrt{6}, -\frac{2\pi}{3}\right)$$

Functions in Polar Coordinates

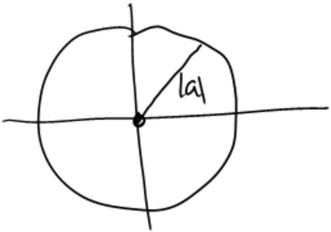
We can relate r and θ like in Cartesian, these are called **polar curves**.

	Cartesian	Polar
Explicit	$y = f(x)$	$r = f(\theta)$
	$y = 3x$	$r = 1 + \sin\theta$
Implicit	$F(x, y) = 0$	$F(r, \theta) = 0$
	$x^2 + y^2 - 1 = 0$	$r\sin\theta = 4$

Example

Problem 8

What curve is represented by the polar equation $r = a$ (where a is a constant)?



$$r = a$$

$$r^2 = a^2$$

$$x^2 + y^2 = a^2$$

Circle centered at the origin with radius $|a|$

Example

Problem 9

Find the polar equation for a circle whose center is the (cartesian) point (a, b) and intersects the origin.

$x^2+y^2=r^2$

$$(x-a)^2 + (y-b)^2 = a^2+b^2$$

$$x^2 - 2ax + \cancel{a^2} + y^2 - 2by + \cancel{b^2} = \cancel{a^2} + \cancel{b^2}$$

$$r^2 - 2ax - 2by = 0 \leftarrow \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}$$

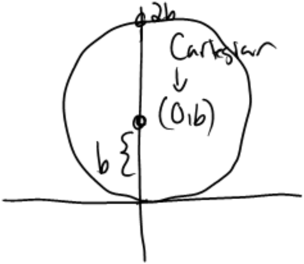
$$r^2 - 2ar\cos\theta - 2br\sin\theta = 0$$

$$r(r - 2a\cos\theta - 2b\sin\theta) = 0 \leftarrow r \neq 0$$

$$r - 2a\cos\theta - 2b\sin\theta = 0, \text{ OR } \boxed{r = 2a\cos\theta + 2b\sin\theta}$$

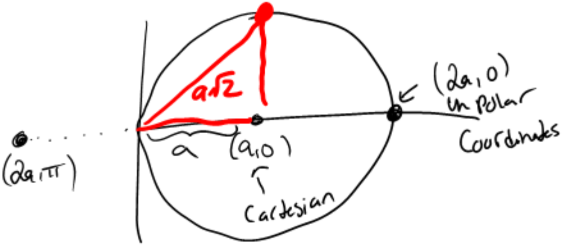
If $a=0$,

$r = 2b\sin\theta$



If $b=0$,

$r = 2a\cos\theta$



θ	r
0	$2a$
$\pi/4$	$2a\left(\frac{\sqrt{2}}{2}\right) = a\sqrt{2}$
$\pi/2$	0
$3\pi/4$	$-a\sqrt{2}$
π	$-2a$

Example

$\theta = \frac{\pi}{2}$ describes a vertical line passing through the origin (when the slope is undefined)

Problem 10

What curve is represented by the polar equation $\theta = \theta_0$ (where θ_0 is a constant)?

In Cartesian coordinates

$$y = \frac{y_0}{x_0} x,$$ where (x_0, y_0) is a non-zero point on the line

When the slope is well defined

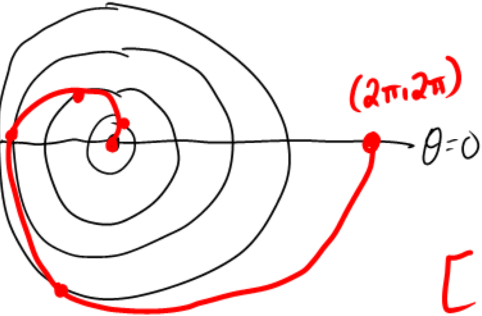
$$y = [\tan \theta_0]\, x$$

Line with a slope of $\tan \theta_0$ passing through the origin

Example

Problem 11

What curve is represented by the polar equation $r = \theta$?



[Archimedian] Spiral

Example

Problem 12

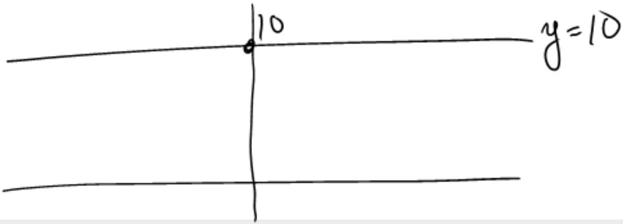
What curve is represented by the polar equation $r \sin\theta = 10$?

Horizontal Line

$y=10$

$$\underbrace{r\sin\theta}_{} = 10$$

$$y = 10$$



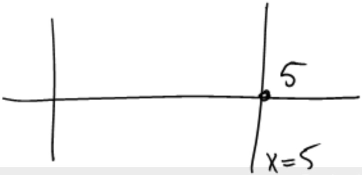
Example

Problem 13

What curve is represented by the polar equation $\underbrace{r\cos\theta}_{x} = 5$?

$x = 5$

The vertical Line $x = 5$

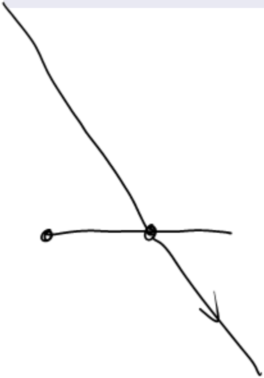


Example

For when the slope is well-defined

Problem 14

Convert the equation for a line $y = mx + b$ into Polar coordinates.



$$y = mx + b$$

$$r\sin\theta = mr\cos\theta + b$$

$$r\sin\theta - rm\cos\theta = b$$

$$r(\sin\theta - m\cos\theta) = b$$

$$\boxed{r = \frac{b}{\sin\theta - m\cos\theta}}$$

$$r = \frac{2}{3\cos\theta + 4\sin\theta}$$; Express this in Cartesian Coordinates

$$= \frac{2}{4\sin\theta + 3\cos\theta} = \frac{2}{4(\sin\theta + \frac{3}{4}\cos\theta)}$$

$$= \frac{1/2}{\sin\theta - \left(-\frac{3}{4}\right)\cos\theta}$$

$$y = -\frac{3}{4}x + \frac{1}{2}$$

Another way: $$r = \frac{2}{3\cos\theta + 4\sin\theta}$$

$$r(3\cos\theta + 4\sin\theta) = 2$$

$$3r\cos\theta + 4r\sin\theta = 2$$

$$3x + 4y = 2$$

$$4y = -3x + 2$$

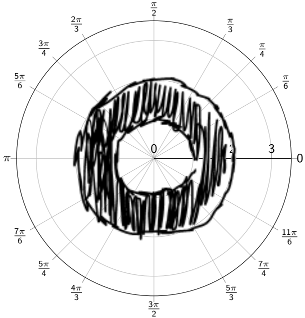
$$y = -\frac{3}{4}x + \frac{2}{4}$$

$$y = -\frac{3}{4}x + \frac{1}{2}$$

Annuli

Problem 15

Describe the region in space: $\boxed{1 \leq r \leq 2}$



$r=1$ $\quad r=2$

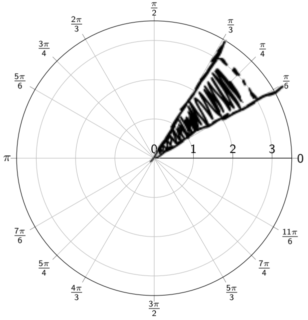
Annulus with inner radius of 1 and an outer radius of 2

Sectors

Problem 16

Describe the region in space: $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}$, $\underbrace{r < 3}$

$0 \leq r < 3$



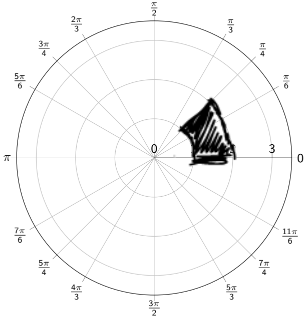
Sector of width

$\frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$

Polar Boxes

Problem 17

Describe the region in space: $1 \leq r \leq 2$, $0 \leq \theta \leq \frac{\pi}{4}$



Polar Box

Polar Rectangle of length 1 and "width" of $\pi/4$