MA 16200: Plane Analytic Geometry and Calculus II

Lecture 31: Calculus in Polar Coordinates

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Sections Covered: 12.3 (Integration, Arc Length)

This part of the notes talks about finding the slope of the tangent line for polar curves. You do not need to know this, but it is added here for completeness.

Derivation

Suppose $r = f(\theta)$, we want to find the slope of the tangent line at θ_0. Unfortunately, it is not as simple as $f'(\theta_0)$.
To find, convert $r = f(\theta)$ into Cartesian coordinates:

$$y = F(x)$$

We now differentiate with respect to θ:

$$\frac{dy}{d\theta} = F'(x)\frac{dx}{d\theta}$$

where $F'(x)$ is the slope of the tangent line. Therefore,

$$F'(x(\theta)) = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

Derivation Continued

We know that:

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(r\cos\theta) = \frac{dr}{d\theta}\cos\theta - r\sin\theta$$
$$\frac{dy}{d\theta} = \frac{d}{d\theta}(r\sin\theta) = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

Putting it all together:

$$F'(\theta) = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Therefore, we now have a way to find the slope of the tangent line as a function of θ.

Example

Problem 1

Find the slopes of the lines tangent to the circle $r = 10$

Sol.

$$F'(\theta) = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{10\cos\theta}{-10\sin\theta} = -\cot\theta$$

Finding Horizontal and Vertical Tangents

Problem 2

Find the points on the interval $-\pi \leq \theta \leq \pi$ at which the cardioid $r = 1 - \cos\theta$ has a vertical or horizontal tangent line.

Sol. $r = 1 - \cos\theta$ and $r' = \sin\theta$. So,

$$F'(\theta) = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\sin^2\theta + (1-\cos\theta)\cos\theta}{\sin\theta\cos\theta - (1-\cos\theta)\sin\theta}$$

$$= -\frac{(2\cos\theta + 1)(\cos\theta - 1)}{\sin\theta(2\cos\theta - 1)}$$

Cont.

Vertical tangent lines occur when $\sin\theta(2\cos\theta - 1) = 0$. On the interval $(-\pi, \pi]$:

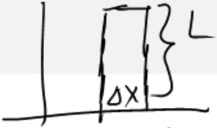
$$\theta \in \{0, -\frac{\pi}{3}, \frac{\pi}{3}, \pi\}$$

Horizontal tangent lines occur when $(2\cos\theta + 1)(\cos\theta - 1) = 0$, but the denominator is **NOT** zero. On the interval $(-\pi, \pi]$:

$$\theta \in \{-\frac{2\pi}{3}, \frac{2\pi}{3}\}$$

There is also a horizontal tangent line at $\theta = 0$, but since $F'(0)$ is in-determinant $(\frac{0}{0})$, that needs to be verified using L'Hôpital's Rule to show $\lim_{\theta\to 0} F'(\theta) = 0$

Area of a Sector



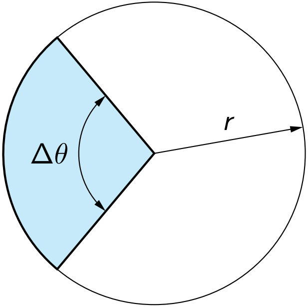
$0 \le y \le L$

Δx

$0 \le r \le R$

$\Delta \theta$

The area of a circular sector of radius r and width $\Delta\theta$ is $A = \frac{1}{2} r^2 \Delta\theta$. Why?



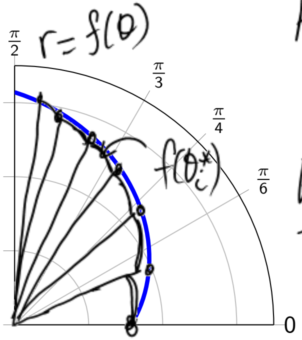
$$\frac{A}{\pi r^2} = \frac{\Delta\theta}{2\pi}$$

$$A = \frac{1}{2} r^2 \Delta\theta$$

Polar Region Area Derivation

How can we find the area of the region below?

Partition $\alpha = \theta_0 < \theta_1 < \theta_2 < \dots < \theta_N = \beta$



Area of each slice:
$\frac{1}{2}[f(\theta_i^*)]^2 \Delta\theta$

$\text{Area} \approx \sum \left(\text{Area of Sectors}\right) = \sum \frac{1}{2}[f(\theta)]^2 \Delta\theta$

Take $N \to \infty$ $(\Delta\theta \to 0)$

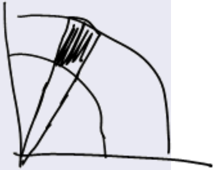
$$\text{Area} = \lim_{N\to\infty} \sum_{i=0}^{N} \frac{1}{2}[f(\theta_i^*)]^2 \Delta\theta = \int_{\alpha}^{\beta} \frac{1}{2}[f(\theta)]^2 d\theta$$

Area Between Polar Curves Formula

Theorem 3

Let R be the regions bounded by the graphs $r = f(\theta)$ and $r = g(\theta)$. between $\theta = \alpha$ and $\theta = \beta$, where f and g are continuous and $f(\theta) \geq g(\theta) \geq 0$ on $[\alpha, \beta]$. The area of R is:

$$A = \int_{\alpha}^{\beta} \frac{1}{2}[f(\theta)^2 - g(\theta)^2] \, d\theta$$



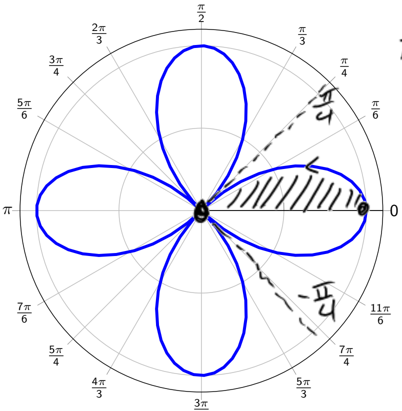
If $g(\theta) = 0$, then:

$$A = \int_{\alpha}^{\beta} \frac{1}{2} f(\theta)^2 \, d\theta = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

Example

Problem 4

Find the area enclosed by one petal of the 4-leaved rose $r = \cos 2\theta$.



$$\text{Area} = 2\left[\begin{matrix}\text{Area of} \\ \tfrac{1}{2}\text{ of a leaf}\end{matrix}\right]$$

$$= 2\int_0^{\pi/4} \frac{1}{2}[\cos 2\theta]^2\, d\theta$$

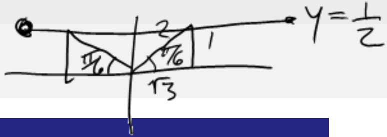
$$= \int_0^{\pi/4} \frac{1}{2}(1+\cos 4\theta)\, d\theta$$

$$= \int_0^{\pi/4} \frac{1}{2}\, d\theta + \frac{1}{4}\cdot\frac{1}{2}\int_0^{\pi/4} \cos 4\theta \; 4d\theta$$

$$= \frac{\pi}{8} + \frac{1}{8}\Big[\sin 4\theta\Big]_0^{\pi/4} = \frac{\pi}{8}$$

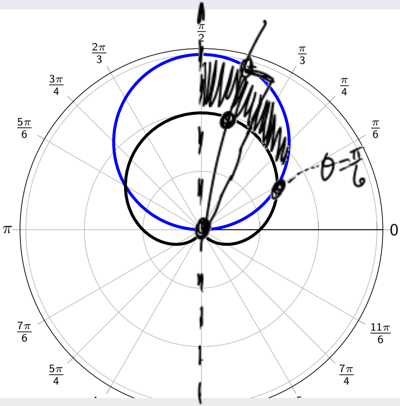
Area of Rose : $4\begin{bmatrix}\text{Area of} \\ \text{a Leaf}\end{bmatrix} = 4\left(\frac{\pi}{8}\right) = \frac{\pi}{2}$

Example



Problem 5

Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$



Find Bounds:

$$3\sin\theta = \sin\theta + 1$$

$$\sin\theta = \frac{1}{2}$$

$$\theta \in \left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$$

$$A = 2\int_{\pi/6}^{\pi/2} \frac{1}{2}\left[(3\sin\theta)^2 - (1+\sin\theta)^2\right] d\theta$$

$$A = \int_{\pi/6}^{\pi/2} \left[9\sin^2\theta - \sin^2\theta - 2\sin\theta - 1 \right] d\theta$$

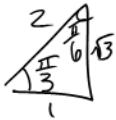
$$= \int_{\pi/6}^{\pi/2} \left[8\sin^2\theta - 2\sin\theta - 1 \right] d\theta$$

$$= \int_{\pi/6}^{\pi/2} \left[4(1 - \cos 2\theta) - 2\sin\theta - 1 \right] d\theta$$

$$= \int_{\pi/6}^{\pi/2} \left[4 - 4\cos 2\theta - 2\sin\theta - 1 \right] d\theta$$

$$= \int_{\pi/6}^{\pi/2} 3\, d\theta - 2\int_{\pi/6}^{\pi/2} \cos 2\theta\, (2)\, d\theta + 2\int_{\pi/6}^{\pi/2} (-\sin\theta)\, d\theta$$

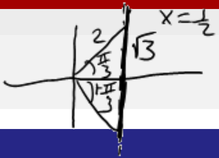
$$= 3\Big[\theta\Big]_{\pi/6}^{\pi/2} - 2\Big[\sin 2\theta\Big]_{\pi/6}^{\pi/2} + 2\Big[\cos\theta\Big]_{\pi/6}^{\pi/2}$$



$$= 3\left(\frac{\pi}{3}\right) - 2\left[0 - \frac{\sqrt{3}}{2}\right] + 2\left[0 - \frac{\sqrt{3}}{2}\right]$$

$$= \pi + \sqrt{3} - \sqrt{3} = \boxed{\pi}.$$

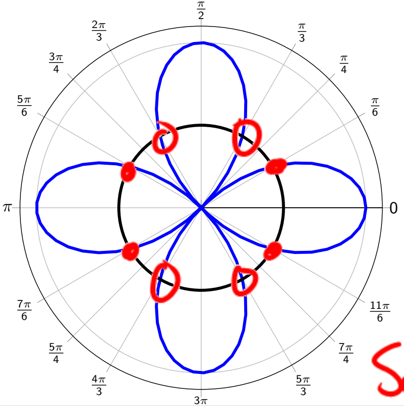
Example



$r=-\frac{1}{2}$

Problem 6

Find all points of intersection of the curves $r = \cos 2\theta$ and $r = \frac{1}{2}$.



$$\cos 2\theta = \frac{1}{2}$$

$$2\theta = \frac{\pi}{3} + 2\pi n \,;\ n \in \mathbb{Z}$$

$$\theta = \frac{\pi}{6} + \pi n$$

OR

$$2\theta = -\frac{\pi}{3} + 2\pi n$$

$$\theta = -\frac{\pi}{6} + \pi n$$

Solving $\cos 2\theta = -\frac{1}{2}$

$x = -\frac{1}{2}$

$$2\theta = \frac{2\pi}{3} + 2\pi n$$

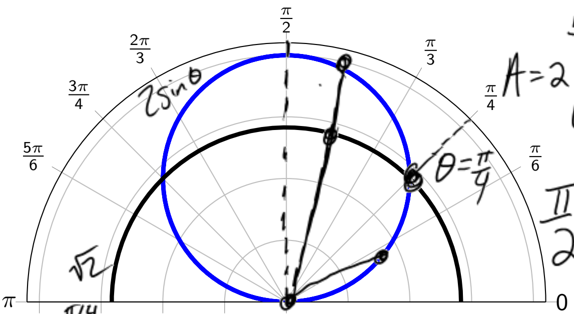
OR

$$2\theta = \frac{4\pi}{3} + 2\pi n$$

Example

Problem 7

Find the area enclosed by the circles $r = \sqrt{2}$ and $r = 2\sin\theta$.



Method 1

$$A = 2\left[\frac{1}{2}\int_0^{\pi/4}(2\sin\theta)^2\,d\theta + \underbrace{\frac{1}{2}\int_{\pi/4}^{\pi/2}(\sqrt{2})^2\,d\theta}_{\pi/4}\right]$$

$$\frac{\pi}{2} + \int_0^{\pi/4} 4\sin^2\theta\,d\theta$$

$$= \frac{\pi}{2} + \int_0^{\pi/4} 2(1-\cos 2\theta)\,d\theta = \frac{\pi}{2} + \int_0^{\pi/4} 2\,d\theta - \int_0^{\pi/4}\cos 2\theta\,(2)\,d\theta$$

$$= \frac{\pi}{2} + \frac{\pi}{2} - \left[\sin 2\theta\right]_0^{\pi/4} = \pi - (1-0) = \pi - 1$$

Method 2

$$A = 2\left(\left[\text{Area of Semi Circle}\right] - \left[\text{Area of the region "outside"}\right]\right.$$

$$A = 2 \cdot \frac{1}{2}\pi(1)^2 - 2\int_{\pi/4}^{\pi/2} \frac{1}{2}\left[(2\sin\theta)^2 - (\sqrt{2})^2\right] d\theta$$

$$= \pi - 1$$

Splitting Regions Into Parts

Problem 8

Given the figure below (from the textbook), set up the integrals to find the area of Regions A, B, C respectively.

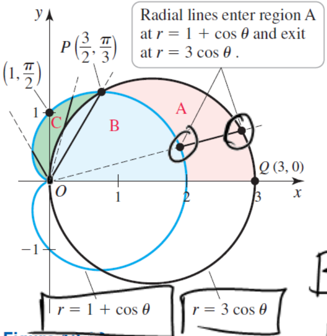


Figure 12.42

A:

$$A = \frac{1}{2}\int_0^{\pi/3} \left[(3\cos\theta)^2 - (1+\cos\theta)^2\right] d\theta$$

$$= \frac{\pi}{2}$$

B:

$$A = \frac{1}{2}\int_0^{\pi/3} (1+\cos\theta)^2 d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2} (3\cos\theta)^2 d\theta$$

$$\begin{pmatrix}\text{Area of}\\ B\end{pmatrix} = \begin{pmatrix}\text{Area of}\\ \text{Semi-Circle}\end{pmatrix} - \begin{pmatrix}\text{Area of}\\ \text{Region } A\end{pmatrix}$$

$$= \frac{1}{2}\pi\left(\frac{3}{2}\right)^2 - \frac{\pi}{2} = \frac{9\pi}{8} - \frac{\pi}{2} = \frac{5\pi}{8}$$

C:

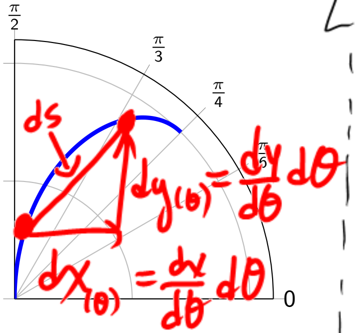
$$A = \frac{1}{2}\int_{\pi/3}^{\pi/2}\left[(1+\cos\theta)^2 - (3\cos\theta)^2\right]d\theta + \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi}(1+\cos\theta)^2\,d\theta$$

$$= \pi/8$$

Arc Length Formula Derivation

$ds = (\qquad) d\theta$

Can we find the length of a curve expressed in polar coordinates when $\theta \in [\alpha, \beta]$?



$L = \int ds$

$$ds = \sqrt{(dx)^2 + (dy)^2}$$
$$= \sqrt{\left(\frac{dx}{d\theta} d\theta\right)^2 + \left(\frac{dy}{d\theta} d\theta\right)^2}$$
$$= \sqrt{\left[\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right][d\theta]^2}$$
$$= \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(r\cos\theta) = \frac{dr}{d\theta}\cos\theta - r\sin\theta$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(r\sin\theta) = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

$$L = \int ds = \int\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta = \int\sqrt{\left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)^2 + (\cdots\cdots)^2}\,d\theta$$

$= \ldots$

$$= \int\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$\frac{dx}{d\theta} \neq r$

$\frac{dr}{d\theta} \neq \frac{dy}{d\theta}$

Arc Length Formula

Theorem 9

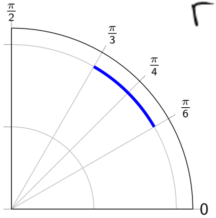
Let $r = f(\theta)$ have a continuous derivative on the interval $[\alpha, \beta]$. The ***arc length*** *of the polar curve $r = f(\theta)$ on $[\alpha, \beta]$ is:*

$$L = \int_{\alpha}^{\beta} \sqrt{f(\theta)^2 + [f'(\theta)]^2}\, d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example

Problem 10

Verify the length of a circular arc from a circle of radius $R > 0$ is $R\Delta\theta$. What is the circumference of the circle?



$r = R \; ; \; \theta_{start} \leq \theta \leq \theta_{end}$

$\Delta\theta = \theta_{end} - \theta_{start}$

$$L = \int_{\theta_{start}}^{\theta_{end}} \sqrt{R^2 + 0}\, d\theta = \int_{\theta_{start}}^{\theta_{end}} R\, d\theta = R\Delta\theta$$

For the full circle, $\Delta\theta = 2\pi$, so $L = 2\pi R$

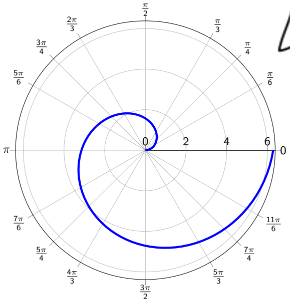
The Archimedean Spiral

$r=\theta$

$\frac{dr}{d\theta}=1$

Problem 11

Find the length of the Archimedean Spiral $r=\theta$ from 0 to some endpoint θ_0.



$$L=\int_0^{\theta_0}\underbrace{\sqrt{\theta^2+1}\,d\theta}_{\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta}\xrightarrow{\theta=\tan\alpha}\int_{\alpha=0}^{\theta_0}\sec^3\alpha\,d\alpha$$

$$=\int_{\alpha=0}^{\theta_0}\sec\alpha\sec^2\alpha\,d\alpha\xrightarrow{\text{By Parts}}\cdots$$

$$=\left[\frac{\theta}{2}\sqrt{\theta^2+1}+\frac{1}{2}\ln\left(\theta+\sqrt{\theta^2+1}\right)\right]_0^{\theta_0}$$

$$=\frac{\theta_0}{2}\sqrt{\theta_0^2+1}+\frac{1}{2}\ln\left(\theta_0+\sqrt{\theta_0^2+1}\right)$$

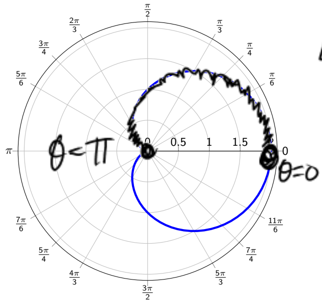
The Cardioid

$r = 1 + \cos\theta$

$\frac{dr}{d\theta} = -\sin\theta$

Problem 12

Find the length of the cardioid $r = 1 + \cos\theta$.



$$L = 2\int_0^{\pi} \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{1 + 2\cos\theta + 1}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{2\left(\frac{1+\cos\theta}{2}\right)\cdot 2}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{4\cos^2\frac{\theta}{2}}\, d\theta$$

$$= 2 \cdot 2\int_0^{\pi} 2\cos\left(\frac{\theta}{2}\right)\left(\frac{1}{2}\right) d\theta = 8\left[\sin\frac{\theta}{2}\right]_0^{\pi}$$

$$= 8[1-0] = 8$$

Example

$$r = e^{2\theta}$$
$$\frac{dr}{d\theta} = 2e^{2\theta}$$

Problem 13

Find the length of the polar curve $r = e^{2\theta}$ from 0 to 2π.

$$L = \int_0^{2\pi} \sqrt{[e^{2\theta}]^2 + [2e^{2\theta}]^2}\, d\theta = \int_0^{2\pi} \sqrt{e^{4\theta} + 4e^{4\theta}}\, d\theta$$
$$= \int_0^{2\pi} \sqrt{5e^{4\theta}}\, d\theta = \frac{1}{2}\int_0^{2\pi} \sqrt{5}\, e^{2\theta}(2)\, d\theta = \frac{\sqrt{5}}{2}\left[e^{2\theta}\right]_0^{2\pi}$$
$$= \frac{\sqrt{5}}{2}\left[e^{4\pi} - 1\right]$$

Example

$$r = \theta^2$$
$$\frac{dr}{d\theta} = 2\theta$$

Problem 14

Find the length of the polar curve $r = \theta^2$ from 0 to 2π.

$$L = \int_0^{2\pi} \sqrt{(\theta^2)^2 + (2\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{\theta^4 + 4\theta^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{\theta^2(\theta^2+4)}\, d\theta = \int_0^{2\pi} \theta\sqrt{\theta^2+4}\, d\theta$$

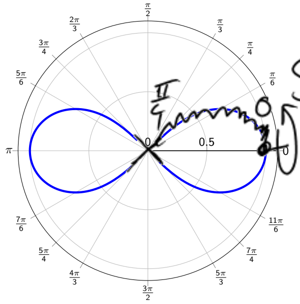
$$= \frac{1}{2}\int_0^{2\pi} \sqrt{\theta^2+4}\,(2\theta)\, d\theta = \frac{1}{2}\cdot\frac{2}{3}(\theta^2+4)^{\frac{3}{2}}\Big]_0^{2\pi} = \frac{1}{3}\left[(4\pi^2+4)^{\frac{3}{2}} - 4^{\frac{3}{2}}\right]$$

Surfaces of Revolution

$r=\sqrt{\cos 2\theta};\ 0\le\theta\le\frac{\pi}{4}$

Problem 15

Find the surface area of the object created when the lemniscate $r^2=\cos 2\theta$ is revolved about the polar axis.



Surface Area Formula?

$$S=\int 2\pi y\,ds=\int 2\pi r\sin\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$r^2=\cos 2\theta$$

$$2r\frac{dr}{d\theta}=-2\sin 2\theta \to \frac{dr}{d\theta}=\frac{-2\sin 2\theta}{2r}=-\frac{\sin 2\theta}{r}$$

$$S = 2\int_0^{\pi/4} 2\pi \underset{\underset{\sqrt{\cos 2\theta}}{\uparrow}}{r} \sin\theta \sqrt{\cos 2\theta + \frac{\sin^2 2\theta}{r^2}}\, d\theta$$

Because we are in a situation we can take the positive root, $r = \sqrt{r^2}$

$r^2 = \cos 2\theta$

$$S = 4\pi \int_0^{\pi/4} \sin\theta \sqrt{r^2\left(\cos 2\theta + \frac{\sin^2 2\theta}{r^2}\right)}\, d\theta \,;$$

$$= 4\pi \int_0^{\pi/4} \sin\theta \sqrt{\cos^2 2\theta + \sin^2 2\theta}\, d\theta$$

$$= 4\pi \int_0^{\pi/4} \sin\theta \, d\theta = 4\pi \left[-\cos\theta\right]_0^{\pi/4}$$

$$= 4\pi\left[-\frac{\sqrt{2}}{2} + 1\right] = \left[4 - 2\sqrt{2}\right]\pi$$