# MA 16200: Plane Analytic Geometry and Calculus II

## Lecture 4: Calculating Volumes & Solids of Revolution

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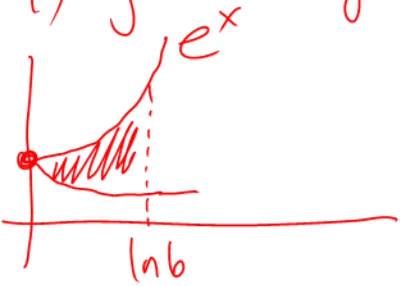
Sections Covered: 6.3

HW

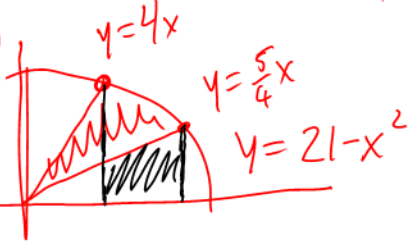
9) $y = e^{x}$; $y = e^{-2x}$; $x = \ln 6$




$$\int_0^{\ln 6} e^{x}\,dx - \int_0^{\ln 6} e^{-2x}\,dx$$

$$= \int_0^{\ln 6} \left[e^{x} - e^{-2x}\right] dx$$

11)




$$\int_0^{m} \left(4x - \tfrac{5}{4}x\right) dx + \int_{m}^{m} \left((21-x^2) - \tfrac{5}{4}x\right) dx$$

# 2ND INTEGRAL

Left Bound $\begin{cases} y = 4x \\ y = 21 - x^2 \end{cases}$ Right Bound $\begin{cases} y = \frac{5}{4}x \\ y = 21 - x^2 \end{cases}$

# Derivation


Volume Slice:

$$V = \begin{pmatrix}\text{Area}\\ \text{of}\\ \text{Base}\end{pmatrix}(\text{Height}) = A(x_i^*)\Delta x$$

Volume Figure:

$$V \approx \sum(\text{Volumes}) \approx \sum_{i=1}^{N} A(x_i^*)\Delta x$$

Take $N \to \infty$ $(\Delta x \to 0)$

$$V = \lim_{N\to\infty} \sum_{i=1}^{N} A(x_i^*)\Delta x = \int_a^b A(x)\,dx$$

# General Slicing Method

## Definition 1

Suppose a solid object extends from $x = a$ to $x = b$, and the cross-sectional area at a point $x$ is given by a function $A(x)$ that can be integrated on $[a, b]$. Then the volume of the solid is:

$$V = \int_a^b A(x)\ dx$$

In "real world", A(x) is usually an integral

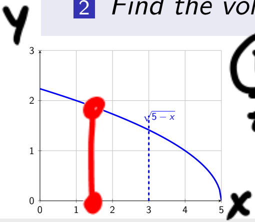
$A(x) = \int z(x,y)\,dy$ ; $V = \int \left( \int z(x,y)\,dy \right) dx$

# Example

## Problem 2

*Consider a solid whose base is the region in the first quadrant bounded by the curve $y = \sqrt{5-x}$ and the line $x = 3$, and whose cross sections through the solid perpendicular to the $x$-axis are squares.*

1. *Find an expression for the cross-sectional area $A(x)$ at a point $x \in [0, 3]$.*
2. *Find the volume of the solid.*




① At slice


$$A(x) = (\text{Base})^2 = \left(\sqrt{5-x}\right)^2 = 5-x$$

# Extra Space

$$\int x^n = \frac{1}{n+1} x^{n+1} \quad (n \neq -1)$$

2 $$V = \int_0^3 (5-x)\,dx = \left[5x - \frac{1}{2}x^2\right]_0^3$$

$$= 5(3) - \frac{1}{2}(9) = 15 - \frac{9}{2} = \boxed{\frac{21}{2}}$$

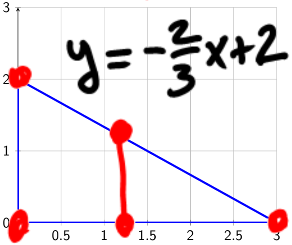
# Extra Space

# Examples

## Problem 3

*Use the general slicing method to find the volume of the solid whose base is the triangle with vertices $(0,0), (3,0)$, and $(0,2)$, and whose cross sections perpendicular to the base and parallel to the y-axis are semicircles*

① Find $A(x)$. At Slice




$y = -\frac{2}{3}x + 2$


Radius of semicircle

$$A = \frac{1}{2}\pi R^2$$

$$= \frac{1}{2}\pi\left(\frac{1}{2}\cdot\left(-\frac{2}{3}x+2\right)\right)^2$$

$$A(x) = \frac{\pi}{8}\left(-\frac{2}{3}x+2\right)^2$$

## Extra Space

$$V = \int_0^3 \frac{\pi}{8}\left(-\frac{2}{3}x+2\right)^2 dx = \frac{\pi}{8}\int_0^3 \left(\frac{4}{9}x^2 + 2\left(-\frac{2}{3}x\right)(2)+4\right) dx$$

$$= \frac{\pi}{8}\int_0^3 \left(\frac{4}{9}x^2 - \frac{8}{3}x + 4\right) dx$$

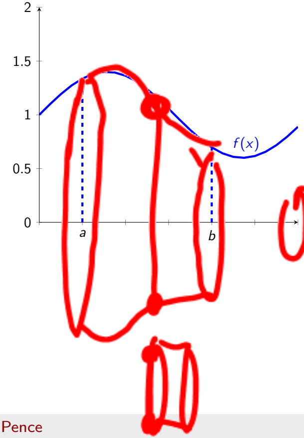
$$= \frac{\pi}{8}\left[\frac{4}{9}\cdot\frac{1}{3}x^3 - \frac{8}{3}\cdot\frac{1}{2}x^2 + 4x\right]_0^3$$

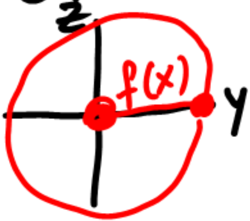
$$= \frac{\pi}{8}\left[\frac{4}{27}\cdot 27 - \boxed{\frac{4}{3}\cdot 9} + 12\right] = \frac{\pi}{2}$$

# Extra Space

# Solids of Revolution

① $A(x)$. At our slice




$$A(x) = \pi(\text{Radius})^2$$
$$= \pi[f(x)]^2$$

$$V = \int_a^b \pi[f(x)]^2\,dx$$

# Disk Method (about the $x$-axis)

## Definition 4

Let $f \geq 0$ be continuous on $[a, b]$. If the region $R$ bounded by the graph of $f$, the $x$-axis, and the lines $x = a$ and $x = b$ is revolved about the $x$-axis, the volume of the resulting solid of revolution is:

$$\boxed{V = \int_a^b \pi[f(x)]^2\, dx}$$

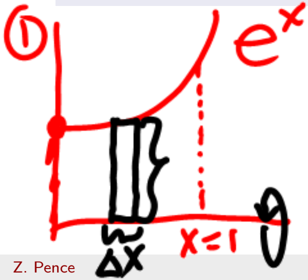
## Examples

### Problem 5

*Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the region and a typical disk.*

Volume (Disk)

1. $y = e^x$; $x = 1$; $y = 0$; $x=0$ *about the $x$-axis.*
2. $y = \frac{1}{x}$; $x = 1$; $x = 2$; $y = 0$; *about the $x$-axis.*




$$V = \pi(\text{Radius})^2(\text{Width})$$
$$= \pi[e^x]^2 \Delta x$$

$$V = \int_0^1 \pi e^{2x}\,dx = \frac{\pi}{2}\int_0^1 e^{2x}\underbrace{(2)\,dx}_{du}$$

$u$

# Extra Space

$$= \frac{\pi}{2}\cdot\left[e^{2x}\right]_0^1 = \frac{\pi}{2}\left[e^2 - 1\right]$$

$u = 2x$

$du = 2dx$

$$\pi\int_0^1 e^{2x}\,dx = \frac{2}{2}\pi\int_0^1 e^{2x}\,dx = \frac{1}{2}\pi\int_0^1 e^{2x}(2)\,dx$$

2


$$V = \int_1^2 \pi\frac{1}{x^2}\,dx = \int_1^2 \pi x^{-2}\,dx = \pi\left[-\frac{1}{x}\right]_1^2 = \pi\left[-\frac{1}{2}+1\right] = \frac{\pi}{2}$$

# Extra Space

# Extra Space

# Extra Space

# Washer Method (about the $x$-axis)

## Definition 6

Let $f$ and $g$ be continuous function with $f(x) \geq g(x) \geq 0$ on $[a, b]$. Let $R$ be the region bounded by $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$. When $R$ is revolved around the $x$-axis, the volume of the resulting solid of revolution is:

$$V = \int_a^b \pi [f(x)]^2 \, dx - \int_a^b \pi [g(x)]^2 \, dx = \int_a^b \pi ([f(x)]^2 - [g(x)]^2) \, dx$$


$$\int_a^b \pi f(x)^2 dx - \int_a^b \pi g(x)^2 dx$$

# Examples

## Problem 7

*Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the region and a typical washer.*

1. $y = \sec x; y = 1; x = -1; x = 1;$ *about the x-axis.*
2. $y = x^2; y^2 = x;$ *about the x-axis.*

1


$$V = \int_{-1}^{1} \pi \sec^2 x \, dx - \int_{-1}^{1} \pi (1) \, dx$$

$$= \pi \int_{-1}^{1} [\sec^2 x - 1] \, dx = \pi \Big[\tan x - x\Big]_{-1}^{1}$$

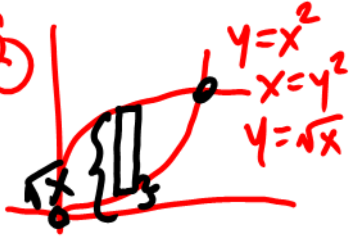
# Extra Space

$$= \pi\left[(\tan 1 - 1) - (\tan(-1) + 1)\right]$$

$$= \pi\left[\tan 1 + \tan(1) - 1 - 1\right]$$

$$= \pi\left[2\tan 1 - 2\right] = 2\pi\left[\tan 1 - 1\right]$$

2




$$V = \int \pi\left((\sqrt{x})^2 - (x^2)^2\right)dx$$

Determine Bounds of Integration $\begin{cases} y=x^2 \\ x=y^2 \end{cases}$

# Extra Space

$$y = (y^2)^2 \Rightarrow y = y^4 \Rightarrow y - y^4 = 0 \Rightarrow y(\underbrace{1-y^3)}_{=0}$$

$$\Rightarrow y = 0, 1 \Rightarrow x = 0, 1$$

$$V = \int_0^1 \pi(x - x^4)\,dx = \pi\left[\tfrac{1}{2}x^2 - \tfrac{1}{5}x^5\right]_0^1$$

$$= \pi\left[\frac{1}{2} - \frac{1}{5}\right] = \frac{3}{10}\pi$$

# Extra Space

# Extra Space

# Rotating about the $y$-axis

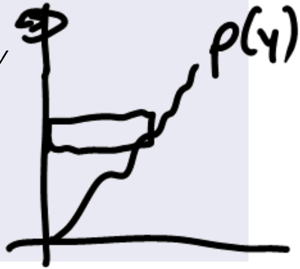
## Definition 8

Let $p$ and $q$ be continuous functions with $p(y) \geq q(y) \geq 0$ on $[c, d]$. Let $R$ be the region bounded by $x = p(y), x = q(y)$, and the lines $y = c$ and $y = d$. When $R$ is revolved about the $y$-axis, the volume of the resulting solid of revolution is given by

$$V = \int_c^d \pi[p(y)^2 - q(y)^2]\, dy$$

If $q(y) = 0$, the disk method results:

$$V = \int_c^d \pi[p(y)]^2\, dy$$



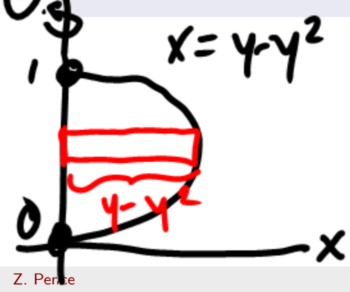
# Examples

## Problem 9

*Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the region and a typical disk/washer.*

1. $x = y - y^2; x = 0$; *about the* $y$*-axis.*
2. $y^2 = x; y = \frac{1}{2}x$; *about the* $y$*-axis.*

①




$$V = \int_0^1 \pi (y-y^2)^2 \, dy$$

$$= \int_0^1 \pi \left[ y^2 - 2y^3 + y^4 \right] dy$$

$$= \pi \left[ \frac{1}{3}y^3 - \frac{1}{2}y^4 + \frac{1}{5}y^5 \right]_0^1$$

# Extra Space

$$V = \pi\left[\frac{1}{3}-\frac{1}{2}+\frac{1}{5}\right] = \pi\left[-\frac{1}{6}+\frac{1}{5}\right] = \boxed{\frac{\pi}{30}}$$

(2)

$y=\frac{1}{2}x \Longleftrightarrow x=2y$

$x=y^2$

Bounds Of Integration

$$\begin{cases} x=2y \\ x=y^2 \end{cases} \Rightarrow y^2=2y \Rightarrow y^2-2y=0$$

$$\Rightarrow y(2-y)=0 \Rightarrow y=0,2$$

$$V=\int_0^2 \pi\left(4y^2-y^4\right)dy = \pi\left[\frac{4}{3}y^3-\frac{1}{5}y^5\right]_0^2$$

# Extra Space

$$= \pi \left[ \frac{4}{3} \cdot 8 - \frac{1}{5} \cdot 32 \right] = \frac{64}{15}\pi$$

Disk/Washers run perpendicular to the axis of rotation

Rotating about $x$-axis $\longleftrightarrow$ Integrate w.r.t $x$

" " $y$-axis $\longleftrightarrow$ Integrate w.r.t $y$

# Extra Space

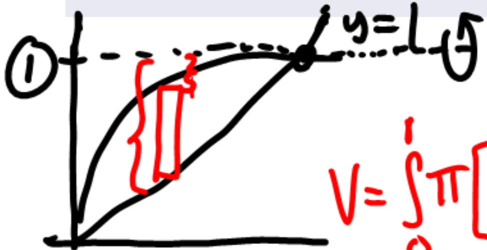
# Extra Space

# Examples

## Problem 10

*Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the region and a typical disk/washer.*

1. $y = x; y = \sqrt{x}$ *about* $y = 1$.
2. $y = \ln x; x = 0$; *on the interval* $0 \leq y \leq 1$; *about* $x = -1$.




Outer Radius: $1-x$

Inner Radius: $1-\sqrt{x}$

$$V=\int_0^1 \pi\left[(1-x)^2-(1-\sqrt{x})^2\right]dx$$

# Extra Space

$$V = \pi \int_0^1 \left[1 - 2x + x^2 - (1 - 2\sqrt{x} + x)\right] dx$$

$$= \pi \int_0^1 \left[1 - 2x + x^2 - 1 + 2\sqrt{x} - x\right] dx$$

$$= \pi \int_0^1 \left[x^2 - 3x + 2x^{\frac{1}{2}}\right] dx$$

$$\frac{x^{\frac{3}{2}}}{\frac{3}{2}}$$

$$= \pi \left[\frac{1}{3}x^3 - \frac{3}{2}x^2 + \frac{4}{3}x^{\frac{3}{2}}\right]_0^1$$

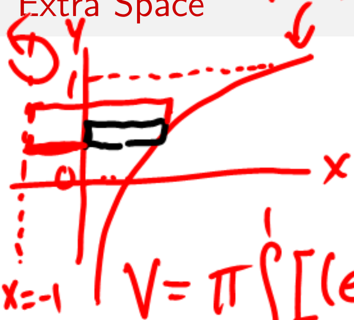
$$\frac{5}{3} - \frac{3}{2}$$

$$\frac{10 - 9}{6}$$

$$= \pi \left[\frac{1}{3} - \frac{3}{2} + \frac{4}{3}\right] = \boxed{\pi/6}$$

# Extra Space

$y = \ln x \longleftrightarrow x = e^y$




Outer Radius: $e^y + 1$

Inner Radius: $1$

$$V = \pi \int_0^1 \left[ (e^y + 1)^2 - (1)^2 \right] dy$$

$$= \pi \int_0^1 \left[ e^{2y} + 2e^y + 1 - 1 \right] dy$$

$$= \pi \int_0^1 \left[ e^{2y} + 2e^y \right] dy$$

# Extra Space

$$= \frac{\pi}{2}\int_0^1 e^{2y}\underbrace{(2)\,dy}_{du} + 2\pi\int_0^1 e^y\,dy$$

$u$

$$= \frac{\pi}{2}\left[e^{2y}\right]_0^1 + 2\pi\left[e^y\right]_0^1$$

$$= \frac{\pi}{2}\left[e^2 - 1\right] + 2\pi\left[e - 1\right]$$

You can stop here

$$= \frac{\pi}{2}\left(e^2 + 4e - 5\right)$$

# Extra Space

# Volume of a Sphere

## Problem 11

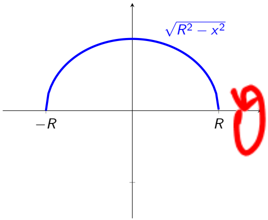
*Show the volume of a sphere of radius $R$ is:*

$$V = \frac{4}{3}\pi R^3$$

$x^2+y^2=R^2$

when $y \geq 0$ $y^2 = R^2 - x^2$

$\Rightarrow y = \sqrt{R^2-x^2}$




Rotate $\sqrt{R^2 - x^2}$ about the $x$-axis, by the disk method:

$$V = \int_{-R}^{R} \pi(\sqrt{R^2 - x^2})^2 \, dx = 2\pi \int_0^R (R^2 - x^2) \, dx$$

$$= 2\pi \left[ R^2 x - \frac{1}{3}x^3 \right]_0^R = 2\pi \left[ \frac{2}{3}R^3 \right] = \frac{4}{3}\pi R^3$$

$$V=\int_{-R}^{R}\pi\left(\sqrt{R^2-x^2}\right)^2dx=\underbrace{\int_{-R}^{R}\pi(R^2-x^2)\,dx}_{f(-x)=f(x)}$$

$$=2\pi\int_{0}^{R}(R^2-x^2)\,dx$$

$$=2\pi\left[R^2x-\frac{1}{3}x^3\right]_0^R=2\pi\left[R^3-\frac{1}{3}R^3\right]$$

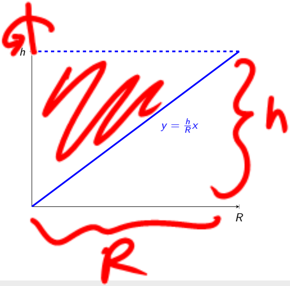
$$=2\pi\left[\frac{2}{3}R^3\right]=\frac{4}{3}\pi R^3$$

# Volume of a Cone

## Problem 12

*Show the volume of a cone of base radius $R$ and height $h$ is:*

$$V = \frac{h}{3}\pi R^2$$




Rotate $x = \frac{R}{h}y$ about the $y$-axis, by the disk method:

$$V = \int_0^h \pi \left(\frac{R}{h}y\right)^2 \, dy = \pi \frac{R^2}{h^2}\left[\frac{1}{3}y^3\right]_0^h$$
$$= \pi \frac{R^2}{h^2}\left[\frac{1}{3}h^3\right] = \frac{h}{3}\pi R^2$$

$$V = \int_0^h \pi \left(\frac{R}{h}y\right)^2 dy = \frac{\pi R^2}{h^2}\int_0^h y^2\,dy$$

$$= \frac{\pi R^2}{h^2}\left[\frac{1}{3}y^3\right]_0^h = \frac{h}{3}\pi R^2$$

Frustum

# Volume of a Wine Barrel

## Problem 13

*The sides of a wine barrel can be approximated by the parabola:*

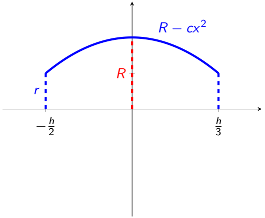
$$y = R - cx^2; \; -\frac{h}{2} \leq x \leq \frac{h}{2}$$

*where $R$ is the maximum radius, $h$ is the height, and $c > 0$ a constant. Show that the volume is:*

$$V = \frac{\pi h}{3}\left[2R^2 + r^2 - \frac{2}{5}(R - r)^2\right]$$

*where $r$ is the minimum radius.*

# Finding the Volume




Rotate $y = R - cx^2$ about the $x$-axis. By the Disk Method:

$$V = \int_{-h/2}^{h/2} \pi(R - cx^2)^2 \, dx = 2\pi \int_0^{h/2} (R - cx^2)^2 \, dx$$

$$= 2\pi \int_0^{h/2} (R^2 - 2Rcx^2 + c^2x^4) \, dx$$

$$= 2\pi \left[ R^2x - \frac{2Rc}{3}x^3 + \frac{c^2}{5}x^5 \right]$$

$$= 2\pi \left[ R^2(h/2) - \frac{2Rc}{3}(h/2)^3 + \frac{c^2}{5}(h/2)^5 \right]$$

$$= \pi R^2 h - \frac{Rc\pi}{6}h^3 + \frac{c^2\pi}{80}h^5$$

## cont.

$$V = \pi h\left(R^2 - \frac{Rc}{6}h^2 + \frac{c^2}{80}h^4\right) = \pi h\left(R^2 - \frac{2R}{3}\left(\frac{ch^2}{4}\right) + \frac{1}{5}\left(\frac{ch^2}{4}\right)^2\right)$$

Let $d = \frac{ch^2}{4} = R - r$:

$$\begin{aligned}
V &= \pi h\left(R^2 - \frac{2R}{3}d + \frac{1}{5}d^2\right) = \frac{\pi h}{3}\left(3R^2 - 2Rd + \frac{3}{5}d^2\right) \\
&= \frac{\pi h}{3}\left(2R^2 + R^2 + d^2 - \frac{2}{5}d^2 - 2Rd\right) \\
&= \frac{\pi h}{3}\left(2R^2 - \frac{2}{5}d^2 + R^2 - 2Rd + d^2\right) = \frac{\pi h}{3}\left(2R^2 - \frac{2}{5}d^2 + (R-d)^2\right) \\
&= \frac{\pi h}{3}\left(2R^2 + r^2 - \frac{2}{5}d^2\right) = \frac{\pi h}{3}\left(2R^2 + r^2 - \frac{2}{5}(R-r)^2\right)
\end{aligned}$$