MA 16200: Plane Analytic Geometry and Calculus II

Lecture 5: The Shell Method

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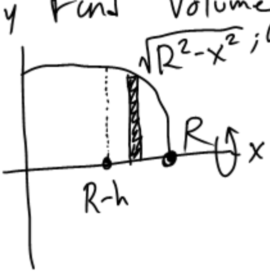
Sections Covered: 6.4

9) Bowl (Hemisphere) Radius R

Filled w/ a depth of h inches from the bottom

Find Volume



$$V = \int_{R-h}^{R} \pi \left(R^2 - x^2\right) dx$$

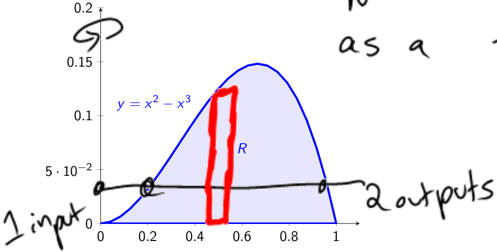
$$= \pi \left[R^2 x - \frac{1}{3}x^3\right]_{R-h}^{R}$$

$$= \pi \left[\left(R^3 - \frac{1}{3}R^3\right) - \left(R^2(R-h) - \frac{1}{3}(R-h)^3\right)\right]$$

The Problem with the Disk/Washer Method

What happens when we try to revolve R about the y-axis and find the volume of the solid of revolution using Disk/Washers?

Need to put $y = x^2 - x^3$ as a function of y.



Can we find a way to revolve around the y-axis, but integrate w.r.t. x?

Volume of a Cylindrical Shell

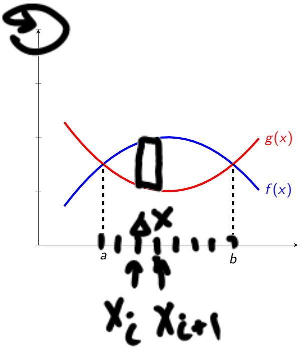
$$V = \pi R^2 h - \pi r^2 h$$

$$= \pi h (R^2 - r^2)$$

$$= \frac{2}{2} \pi h (R+r)(R-r)$$

$$= \underbrace{2\pi\left(\frac{R+r}{2}\right)}_{\text{Average Circumference}} \underset{\text{height}}{h} \underbrace{(R-r)}_{\text{Thickness}}$$

Shell Method Derivation



Volume Shell:

$$V = (\text{Circumference})(\text{Height})(\text{Width})$$

$$= \Big(2\pi \underbrace{\bar{x}_i}_{\frac{x_i + x_{i+1}}{2}}\Big)\big(f(x_i) - g(x_i)\big)\Delta x$$

Volume

$$V \approx \sum \begin{pmatrix}\text{Volumes}\\ \text{Shells}\end{pmatrix}$$

As $\Delta x \to 0$, $V = \int_a^b 2\pi x\big(f(x) - g(x)\big)\,dx$

The Shell Method

Definition 1

Let f and g be continuous functions with $f(x) \geq g(x)$ on $[a, b]$. If R is the region bounded by the curves $y = f(x)$ and $y = g(x)$ between the lines $x = a$ and $x = b$, the volume of the solid generated when R is revolved about the y-axis is:

$$V = \int_a^b \overbrace{2\pi x}^{\text{Circumference}} \underbrace{(f(x) - g(x))}_{\text{Height}} \underbrace{dx}_{\text{Thickness}}$$

← Analog to Washer Method

When R is bounded by the x-axis ($g(x) \equiv 0$), then

$$V = \int_a^b 2\pi x \, f(x) \, dx$$

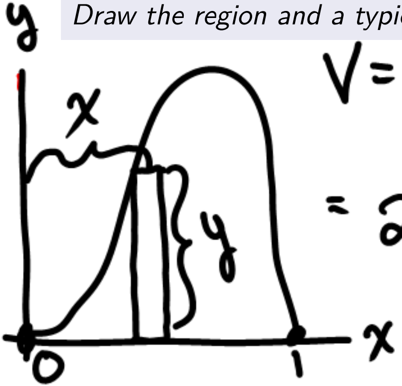
← Analog to Disk Method

Example

Intersects x-axis at $x = 0, 1$

Problem 2

Let R be the region bounded by $y = x^2 - x^3$ and the x-axis. Find the volume of the solid when R is revolved around the y-axis. Draw the region and a typical shell.



$$V = \int_0 2\pi x (x^2 - x^3)\,dx$$

$$= 2\pi \int_0^1 (x^3 - x^4)\,dx$$

$$= 2\pi \left[\frac{1}{4}x^4 - \frac{1}{5}x^5 \right]_0^1$$

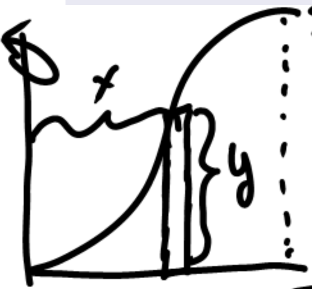
Extra Space

$$V = 2\pi\left[\frac{1}{4} - \frac{1}{5}\right] = 2\pi\left[\frac{1}{20}\right] = \frac{\pi}{10}$$

Example

Problem 3

Let R be the region bounded by $f(x) = \sin x^2$, $x = \sqrt{\pi/2}$, and the x-axis. Find the volume of the solid when R is revolved around the y-axis. Draw the region and a typical shell.



$x = \sqrt{\frac{\pi}{2}}$

$$V = \int_0^{\sqrt{\pi/2}} 2\pi x (\sin x^2)\,dx$$

$$= \pi \int_0^{\sqrt{\pi/2}} \sin \overbrace{x^2}^{u}\ \overbrace{(2x)\,dx}^{du}$$

$$= \pi \left[-\cos x^2\right]_0^{\sqrt{\pi/2}}$$

Extra Space

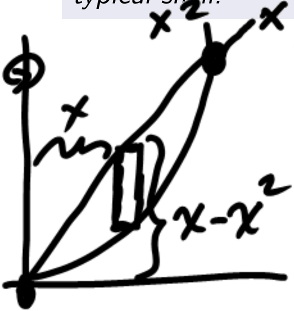
$$= \pi\left[0 - (-1)\right] = \pi$$

Example

Bounds solve $\begin{cases} y=x \\ y=x^2 \end{cases} \to x=0,1$

Problem 4

Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$. Draw the region and a typical shell.



$$V = \int_0^1 2\pi x(x-x^2)\,dx$$

$$= 2\pi \int_0^1 (x^2-x^3)\,dx$$

$$= 2\pi\left[\tfrac{1}{3}x^3 - \tfrac{1}{4}x^4\right]_0^1$$

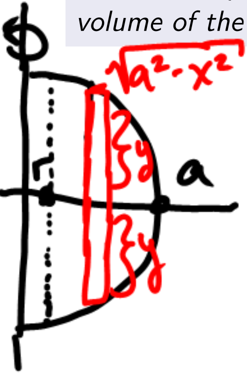
Extra Space

$$= 2\pi\left[\frac{1}{3} - \frac{1}{4}\right] = 2\pi\left[\frac{1}{12}\right] = \pi/6$$

Example

Problem 5

A cylindrical hole with radius r is drilled symmetrically through the center of a sphere with radius a, where $0 \leq r \leq a$. What is the volume of the remaining material?



Radius: x

Height: $\sqrt{a^2-x^2}+\sqrt{a^2-x^2}=2\sqrt{a^2-x^2}$

$$V=\int_r^a 2\pi x\left(2\sqrt{a^2-x^2}\right)dx \quad \left\{\begin{array}{l} u=a^2-x^2 \\ du= \\ -2x\,dx \end{array}\right.$$

Extra Space

$$= (-1)2\pi \int_r^a \underbrace{\sqrt{a^2-x^2}}_{u} \, \underbrace{((-1)2x)\,dx}_{du}$$

$$= -2\pi \frac{2}{3}\left[(a^2-x^2)^{\frac{3}{2}}\right]_r^a$$

$$= -\frac{4\pi}{3}\left[0 - (a^2-r^2)^{\frac{3}{2}}\right] = \frac{4\pi}{3}(a^2-r^2)^{\frac{3}{2}}$$

Formulas for Rotating about the x-axis

Definition 6

Let p and q be continuous functions with $p(y) \geq q(y)$ on $[c, d]$. If R is the region bounded by the curves $x = p(y)$ and $x = q(y)$ between the lines $y = c$ and $y = d$, the volume of the solid generated when R is revolved about the x-axis is:

$$V = \int_c^d 2\pi y(p(y) - q(y))\, dy$$

When R is bounded by the y-axis ($q(y) \equiv 0$), then

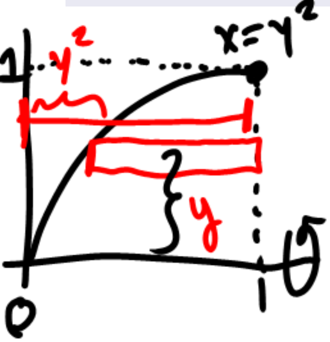
$$V = \int_c^d 2\pi y p(y)\, dy$$

Example

(☆ Shells run parallel to the axis of Rotation)

Problem 7

Use cylindrical shells to find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1. Draw the region and a typical shell.



Radius: y

Height: $1-x = 1-y^2$

$$V = \int_0^1 2\pi y(1-y^2)\,dy$$

$$= 2\pi \int_0^1 (y-y^3)\,dy$$

Extra Space

$$V = 2\pi\left[\frac{1}{2}y^2 - \frac{1}{4}y^4\right]_0^1 = 2\pi\left[\frac{1}{2} - \frac{1}{4}\right]$$

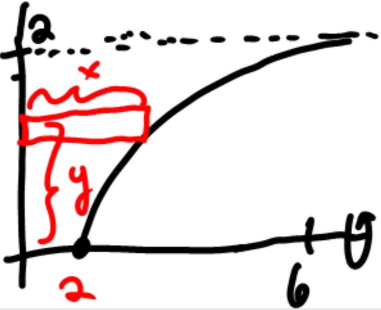
$$= 2\pi\left[\frac{1}{4}\right] = \pi/2$$

Example

$x = y^2 + 2$

Problem 8

Let R be the region in the first quadrant bounded by the graph of $y = \sqrt{x-2}$ and the line $y = 2$. Find the volume of the solid generated when R is revolved about the x-axis. Draw the region and a typical shell.



Radius: y

Height: $x = y^2 + 2$

$$V = \int_0^2 2\pi y (y^2+2)\,dy$$

$$= 2\pi \int_0^2 (y^3 + 2y)\,dy$$

Extra Space

$$V = 2\pi\left[\frac{1}{4}y^4 + y^2\right]_0^2 = 2\pi\left[\frac{1}{4}(2)^4 + (2)^2\right]$$

$$= 2\pi[4+4] = 16\pi$$

Example

Problem 9

Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the x-axis. Draw the region and a typical shell.

$$V = \int_0^8 2\pi y \cdot y^{\frac{1}{3}}\,dy$$

$$= 2\pi \int_0^8 y^{\frac{4}{3}}\,dy = 2\pi \cdot \left[\frac{3}{7} y^{\frac{7}{3}}\right]_0^8$$

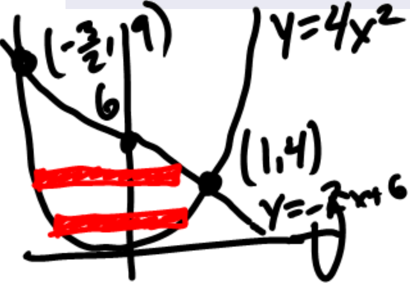
$$= \frac{256 \cdot 3}{7}\pi = \frac{768}{7}\pi$$

Extra Space

Example

Problem 10

Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by $y = 4x^2$ and $2x + y = 6$ about the x-axis. Draw the region and a typical shell.



① Find where to split the integral

$$\begin{cases} y=4x^2 \\ y=-2x+6 \end{cases} \Rightarrow -2x+6=4x^2$$

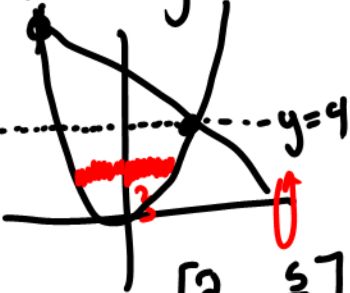
$$\Rightarrow 4x^2+2x-6=0 \Rightarrow (2x+3)(2x-2) = 0$$

Extra Space

$y=4x^2$
$\Rightarrow x=\frac{1}{2}\sqrt{y}$

$$\Rightarrow x=-\frac{3}{2},1 \Rightarrow y=9,4$$

1st Integral



Radius: y

Height: $\frac{\sqrt{y}}{2}+\frac{\sqrt{y}}{2}=\sqrt{y}$

$$V=\int_0^4 2\pi y\sqrt{y}\,dy=2\pi\int_0^4 y^{\frac{3}{2}}\,dy$$

$$=2\pi\cdot\left[\frac{2}{5}y^{\frac{5}{2}}\right]_0^4=2\pi\left(\frac{2}{5}\right)\underbrace{(4)^{\frac{5}{2}}}_{2^5=32}=\frac{128\pi}{5}$$

2ND INTEGRAL

Height: $\frac{\sqrt{y}}{2} + \left(-\frac{1}{2}y+3\right)$

Radius: y

$$V=\int_4^9 2\pi y\left(\frac{1}{2}\sqrt{y}-\frac{1}{2}y+3\right)dy$$

$$=\pi\int_4^9 y\left(\sqrt{y}-y+6\right)dy=\pi\int_4^9\left(y^{\frac{3}{2}}-y^2+6y\right)dy$$

$$=\pi\left[\frac{2}{5}y^{\frac{5}{2}}-\frac{1}{3}y^3+3y^2\right]_4^9=\frac{866\pi}{15}$$

Total Volume

$$\frac{128}{5}\pi+\frac{866}{15}\pi$$

$$=\frac{250}{3}\pi$$

Example

$x(1-x) \overset{\text{set}}{=} 0 \Rightarrow x = 0, 1$

Problem 11

Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.

Height: $y = x - x^2$

Radius: $2 - x$

$$V = \int_0^1 2\pi(2-x)(x-x^2)\,dx$$

$$= 2\pi \int_0^1 (2x - 2x^2 - x^2 + x^3)\,dx$$

Extra Space

$$V = 2\pi \int_0^1 (2x - 3x^2 + x^3)\,dx = 2\pi \left[x^2 - x^3 + \frac{1}{4}x^4 \right]_0^1$$

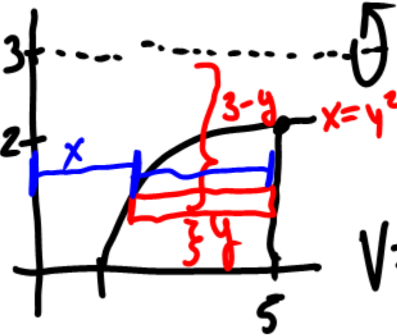
$$= 2\pi \left[1 - 1 + \frac{1}{4} \right] = 2\pi \left[\frac{1}{4} \right] = \frac{\pi}{2}$$

Example

Problem 12

$x = y^2 + 1$

Let R be the region bounded by the curve $y = \sqrt{x-1}$, the line $y = 0$, and $x = 5$. Use the shell method to find the volume of the solid generated when R is revolved about the line $y = 3$.



$x = y^2 + 1$

Radius: $3 - y$

Height: $5 - x = 5 - (y^2 + 1)$
$= 4 - y^2$

$$V = \int_0^2 2\pi(3-y)(4-y^2)\,dy$$

Extra Space

$$V = 2\pi \int_0^2 (12 - 3y^2 - 4y + y^3)\,dy$$

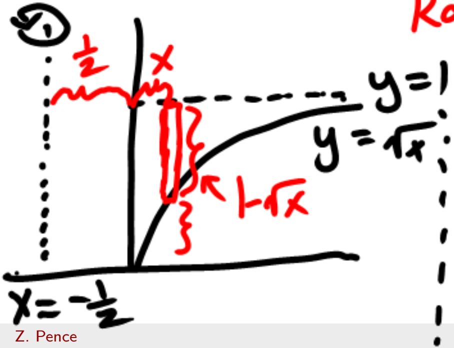
$$= 2\pi \left[12y - y^3 - 2y^2 + \frac{1}{4}y^4\right]_0^2$$

$$= 2\pi\left[24 - 8 - 8 + 4\right] = 24\pi$$

Example (if time allows)

Problem 13

Let R be the region bounded by the curve $y = \sqrt{x}$, the line $y = 1$, and the y-axis. Use the shell method to find the volume of the solid generated when R is revolved about the line $x = -\frac{1}{2}$.



Radius: $x + \frac{1}{2}$

Height: $1 - \sqrt{x}$

$$V = \int_0^1 2\pi \left(x+\frac{1}{2}\right)(1-\sqrt{x})\,dx$$

Extra Space

$$V = 2\pi \int_0^1 \left(x - x^{\frac{3}{2}} + \frac{1}{2} - \frac{1}{2} x^{\frac{1}{2}}\right) dx$$

$$= 2\pi \left[\frac{1}{2}x^2 - \frac{2}{5}x^{\frac{5}{2}} + \frac{1}{2}x - \frac{1}{2}\cdot\frac{2}{3}x^{\frac{3}{2}}\right]_0^1$$

$$= 2\pi\left[\frac{1}{2} - \frac{2}{5} + \frac{1}{2} - \frac{1}{3}\right] = 2\pi\left[\frac{3}{5} - \frac{1}{3}\right]$$

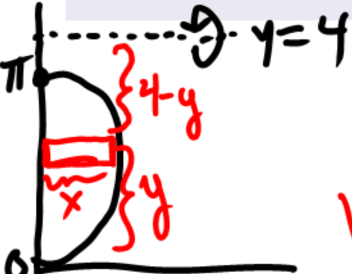
$$= 2\pi\left[\frac{4}{15}\right] = \frac{8}{15}\pi$$

Example (if time allows)

Problem 14

Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified axis.

$$x = \sqrt{\sin y}; \quad 0 \le y \le \pi; \quad x = 0; \quad \textit{about } y{=}4$$



Height: $\sqrt{\sin y}$

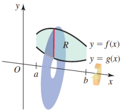
Radius: $4-y$

$$V = \int_0^{\pi} 2\pi(4-y)\sqrt{\sin y}\,dy$$

Extra Space

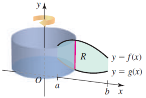
Summary

Integration with respect to x



Disk/washer method about the x-axis
Disks/washers are *perpendicular* to the x-axis.

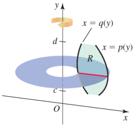
$$\int_a^b \pi\big(\underbrace{f(x)}_{\text{outer radius}}{}^2 - \underbrace{g(x)}_{\text{inner radius}}{}^2\big)\,dx$$



Shell method about the y-axis
Shells are *parallel* to the y-axis.

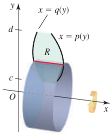
$$\int_a^b \underbrace{2\pi x}_{\text{shell circumference}}\underbrace{(f(x) - g(x))}_{\text{shell height}}\,dx$$

Integration with respect to y



Disk/washer method about the y-axis
Disks/washers are *perpendicular* to the y-axis.

$$\int_c^d \pi\big(\underbrace{p(y)}_{\text{outer radius}}{}^2 - \underbrace{q(y)}_{\text{inner radius}}{}^2\big)\,dy$$



Shell method about the x-axis
Shells are *parallel* to the x-axis.

$$\int_c^d \underbrace{2\pi y}_{\text{shell circumference}}\underbrace{(p(y) - q(y))}_{\text{shell height}}\,dy$$