# MA 16200: Plane Analytic Geometry and Calculus II

## Lecture 7: Mass and Work Problems

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Sections Covered: 6.7 (Up to Lifting Problems)

Arc Length: $y = 1 - x^2$ ; $[-1, 1]$

$f'(x) = -2x$

$[f'(x)]^2 = 4x^2$

$$L = \int_{-1}^{1} \sqrt{1+4x^2}\,dx = \int_{-1}^{1} \sqrt{1+(2x)^2}\,dx$$ ; Let $u = 2x$

$du = 2\,dx$

$dx = \frac{1}{2}du$

# Mass Formula Derivation

*Mass* = (*Density*)(*Volume*), but what happens when the density is non-constant?


$$\text{Mass (Segment)}: (\text{Density})(\text{Volume}) = \underbrace{\overset{\text{rho}}{\rho}(x_i^*)}_{\text{Density Function}}\,\Delta x$$

$$\text{Mass} \approx \sum \begin{pmatrix}\text{Density}\\ \text{Segments}\end{pmatrix} \approx \sum_i \rho(x_i^*)\,\Delta x$$

Take $\Delta x \to 0$, $\text{Mass} = \int_a^b \rho(x)\,dx$

# Mass of a 1-D Object

**Definition 1**

Suppose a thin bar or wire is represented by the interval $a \leq x \leq b$ with a density function $\rho = \rho(x)$ (with units of mass per length). The **mass** of the object is:

$$m = \int_a^b \rho(x)\ dx$$

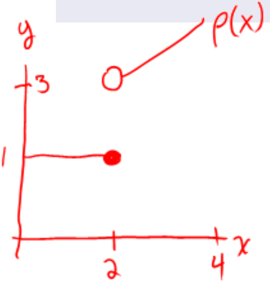
Note: Assume all units are SI units unless otherwise stated

# Example

## Problem 2

*Find the mass of the thin rod given the density function:*

$$\rho(x) = \begin{cases} 1 & \text{if } 0 \le x \le 2 \\ 1+x & \text{if } 2 < x \le 4 \end{cases} \quad \frac{\text{kg}}{\text{m}}$$




$$m = \int_0^4 \rho(x)\,dx = \int_0^2 \rho(x)\,dx + \int_2^4 \rho(x)\,dx$$

$$= \int_0^2 (1)\,dx + \int_2^4 \overbrace{(1+x)}^{u=x+1}\,dx = 2 + \left[\frac{1}{2}(1+x)^2\right]_2^4$$

$$= 2 + \left[\frac{5^2}{2} - \frac{3^2}{2}\right] = 2 + \left[\frac{\overbrace{25-9}^{16}}{2}\right] = 10\ \text{kg}$$

# Extra Space

# Example

## Problem 3

*A thin, two-meter bar, represented by the interval* $0 \leq x \leq 2$*, is made of an alloy whose density in units of kg/m is given by* $\rho(x) = 1 + x^2$*. What is the mass of the bar?*


$$m = \int_0^2 \rho(x)\,dx = \int_0^2 (1+x^2)\,dx$$

$$= \left[x + \frac{1}{3}x^3\right]_0^2 = 2 + \frac{1}{3}\cdot 8$$

$$= 2 + \frac{8}{3} = \frac{6}{3} + \frac{8}{3} = \frac{14}{3}\text{ kg}$$

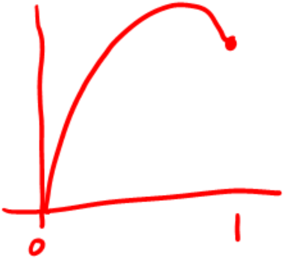
# Extra Space

# Example

## Problem 4

*Find the mass of the thin rod given the density function:*

$$\rho(x) = x\sqrt{2-x^2}; \quad 0 \leq x \leq 1$$




$$m = \int_0^1 x\sqrt{2-x^2}\,dx$$

$$= -\frac{1}{2}\int_0^1 \underbrace{\sqrt{2-x^2}}_{u=2-x^2}(-2x)\,dx$$

$$= -\frac{1}{2}\cdot\frac{2}{3}\left[(2-x^2)^{\frac{3}{2}}\right]_0^1$$

# Extra Space

$$M = -\frac{1}{3}\left[1 - 2^{\frac{3}{2}}\right] = -\frac{1}{3}\left[1 - \sqrt{8}\right] = \frac{1}{3}\left[\sqrt{8} - 1\right]$$

$$= \frac{1}{3}\left[2\sqrt{2} - 1\right] \text{ kg}$$

$$\rho(x) = \text{kg}/\text{m}$$

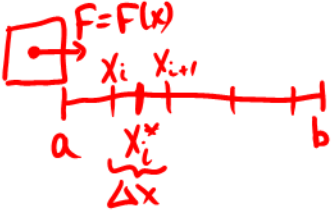
$$x = \text{m}$$

$$\int \left(\frac{\text{kg}}{\text{m}}\right) \text{m}$$

# Derivation

$W = \vec{F} \cdot \vec{D}$

*Work* = (*Force*)(*Distance*) $\cos\theta$, but what happens when the force is non-constant (assuming the object is moving in a straight line and $\theta = 0$).




Work (Segment): (Force)(Distance)

$\approx F(x_i^*)\Delta x$

$$\text{Work} \approx \sum \left(\begin{matrix}\text{Work}\\ \text{over}\\ \text{segments}\end{matrix}\right) = \sum_i F(x_i^*)\Delta x$$

Take $\Delta x \to 0$, $W = \int_a^b F(x)\,dx$

# Work

**Definition 5**

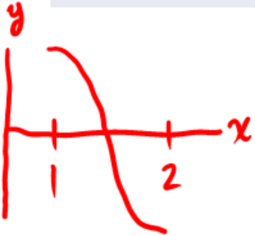
The work done by a variable force $F = F(x)$ moving an object along the line $x = a$ to $x = b$ in the direction of the force is:

$$W = \int_a^b F(x)\, dx$$

# Example

## Problem 6

*When a particle is located a distance $x$ meters from the origin, a force of $\cos\frac{\pi}{3}x$ Newtons acts on it. How much work is done in moving the particle from $x = 1$ to $x = 2$?*

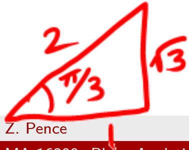



$$W = \int_1^2 \cos\frac{\pi}{3}x\,dx = \frac{3}{\pi}\int_1^2 \cos\left(\frac{\pi}{3}x\right)\left[\frac{\pi}{3}\right]dx$$

$$= \frac{3}{\pi}\left[\sin\left(\frac{\pi}{3}x\right)\right]_1^2 = \frac{3}{\pi}\left[\sin\left(\frac{2\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right)\right]$$

$$= \frac{3}{\pi}\left[\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right]$$

$$= 0\ \mathrm{N\cdot m} = 0\ \underbrace{\mathrm{J}}_{\text{Joules}}$$

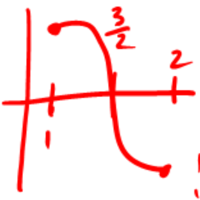



# Extra Space

# Example

## Problem 7

*Interpret your answer to the previous problem by considering the work done from $x = 1$ to $x = \frac{3}{2}$ and from $x = \frac{3}{2}$ to $x = 2$.*




$$W = \int_1^{3/2} \cos\left(\frac{\pi}{3}x\right)dx = \frac{3}{\pi}\left[\sin\left(\frac{\pi}{3}x\right)\right]_1^{3/2}$$

$$= \frac{3}{\pi}\left[\sin\frac{\pi}{2} - \sin\frac{\pi}{3}\right] = \frac{3}{\pi}\left[1 - \frac{\sqrt{3}}{2}\right]$$

$$W = \int_{3/2}^{2} \cos\left(\pi/3\, x\right)dx = \frac{3}{\pi}\left[\sin\left(\frac{\pi}{3}x\right)\right]_{3/2}^{2}$$

$$= \frac{3}{\pi}\left[\sin\left(\frac{2\pi}{3}\right) - \sin\frac{\pi}{2}\right] = \frac{3}{\pi}\left[\frac{\sqrt{3}}{2} - 1\right] = -\frac{3}{\pi}\left[1 - \frac{\sqrt{3}}{2}\right]$$

# Extra Space

# Example

## Problem 8

*Newton's Law of Gravitation states that two bodies with masses $m_1$ and $m_2$ attract each other with a force:*

$$F(r) = G\frac{m_1 m_2}{r^2}$$


*where $r$ is the distance between the bodies and $G$ is the gravitational constant. If one of the bodies is fixed, find the work needed to move the other from $r = a$ to $r = b$.*

$$W = \int_a^b G\frac{m_1 m_2}{r^2}\,dr = G m_1 m_2 \int_a^b \frac{1}{r^2}\,dr = G m_1 m_2 \left[-\frac{1}{r}\right]_a^b$$

$$= G m_1 m_2 \left[-\frac{1}{b} + \frac{1}{a}\right] = G m_1 m_2 \left[\frac{1}{a} - \frac{1}{b}\right]$$

# Extra Space

# Example

## Problem 9

*Use Newton's Law of Gravitation to compute the work required to launch a 1000-kg satellite vertically to an orbit ~~1000 km~~ $10^6$ m high. You may assume that Earth's mass is $M = 5.98 \times 10^{24}$ kg and is concentrated at its center. Take the radius of $R = 6.37 \times 10^6 m$ and $G = 6.67 \times 10^{-11}\ N \cdot m^2/kg^2$.*

$$W = \int_R^{R+10^6} GM(1000)\frac{1}{r^2}\,dr = GM(1000)\left[\frac{1}{R} - \frac{1}{R+10^6}\right]$$


$$= (6.67\times 10^{-11})(5.98\times 10^{24})(1000)\left[\frac{1}{6.37\times 10^6} - \frac{1}{\underbrace{(6.37\times 10^6)+10^6}}\right]$$

in the denom

# Extra Space

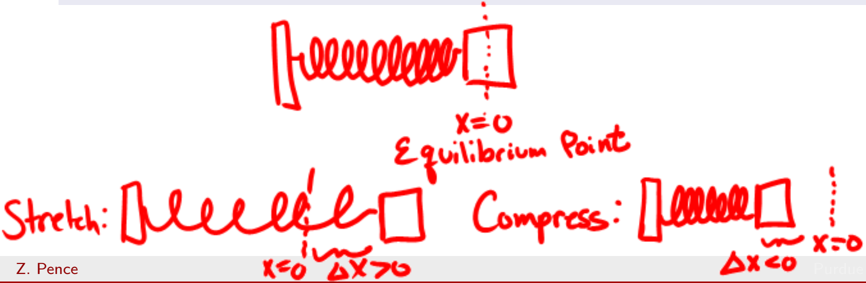
# Hooke's Law

## Definition 10 (Hooke's Law)

The force required to keep a spring stretched/compressed is directly proportional to the displacement from its equilibrium position. In symbols,

$$F(x) = kx$$

where $k$ is called the **spring constant**.



# Example

## Problem 11

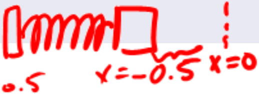
*Suppose a force of 10 N is required to stretch a spring 0.1 m from its equilibrium position and hold it in that position.*
*(a) Assuming the spring obeys Hooke's law, find the spring constant $k$.*

$$F(x) = kx$$
$$10 = k(0.1)$$
$$k = 100 \frac{N}{m}$$

# Example (cont.)

## Problem 12

*(b) How much work is needed to **compress** the spring 0.5 m from its equilibrium position?*



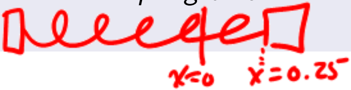

$$W = \int_0^{-0.5} F(x)\,dx = \int_0^{-0.5} (100)x\,dx = 50x^2\Big]_0^{-0.5}$$

$$= 50(0.25) = 12.5 \text{ Nm} = 12.5 \text{ J}$$

# Example (cont.)

## Problem 13

*(c) How much work is needed to* ***stretch*** *the spring 0.25 m from its equilibrium position? position?*




$$W = \int F(x)\,dx = \int_0^{0.25} 100x\,dx$$

$$= 50x^2\Big]_0^{0.25} = \frac{50}{16} = 3.125\text{ J}$$

# Example (cont.)

## Problem 14

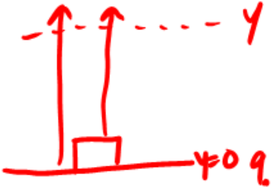
*(d) How much additional work is required to stretch the spring 0.25 m if it has already been stretched 0.1 m from its equilibrium position?*


$$W = \int_{0.1}^{0.25} F(x)\,dx = \int_{0.1}^{0.25} 100x\,dx = 50x^2\Big]_{0.1}^{0.25}$$

$$= \frac{50}{16} - \frac{8}{16} = \frac{42}{16} = 2.625\text{ J}$$

# Work caused by gravity

What is the work required to lift an object of mass $m$ a distance $y$ upward?




Work = (Force) (Distance)

$= mg\Delta y$

If the object in on the ground
Initial y is 0

$W = mgy$ —— (*)

Force:

$F = m\frac{d^2x}{dt^2} = mg$

$9.8\frac{m}{s^2}$

Acceleration Due to Gravity

# Lifting a Chain

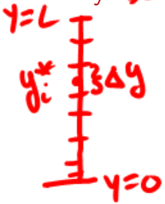
## Definition 15

The work required to lift a chain of density $\rho$ hanging vertically from $y = 0$ to $y = L$ is:

$$W = \int_0^L \rho g(L - y)\, dy$$

Why? Density $\rho$ ; Work(Segments) $= mg(L - y_i^*)$

$= \rho g(L - y_i^*)\Delta y$




In the limit,

$$W = \int_0^L \rho g(L-y)\, dy$$

# Example

## Problem 16

*A ten-meter chain with* $\overbrace{\textit{density of 1.5 kg/m}}^{\rho}$ *hangs from a platform at a construction site that is 11 meters above the ground. Compute the work required to lift the chain to the platform.*


If y=0 represents the ground

$$W=\int_1^{11}\rho g(11-y)\,dy$$

If y=0 represents the base of the chain

$$W=\int_0^{10}\rho g(10-y)\,dy=(-1)\int_0^{10}\rho g(10-y)[(-1)]\,dy$$

## Extra Space

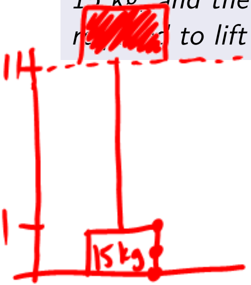
$$= -\rho g \cdot \frac{1}{2}\Big[(10-y)^2\Big]_0^{10} = -\frac{\rho g}{2}\big[0-100\big] = 50\ \rho g$$

$$= 50\ (1.5)(9.8) = 735\ \text{J}$$

# Example

## Problem 17

*Several packages of nails are placed in a one-meter-tall bucket that rests on the ground; the mass of the bucket and nails together is 15 kg, and the chain is attached to the bucket. How much work is required to lift the bucket to the platform?*




Work Chain: 735 J

$$\int_0^1 mg(11)\,dy$$

Work (Bucket): $mg\Delta y = 11mg$

$= 11(15)(9.8) \approx 1617\text{ J}$

Total Work: $735 + 1617 = 2352\text{ J}$

# Example

## Problem 18

*A cable that weighs 2 lb/ft is used to lift 800lbs of coal up a mineshaft 500 ft deep. Find the work done.*

Force (Metric): $F = mg$ in Newtons

US: Weight $= \left(\frac{\text{weight}}{\text{ft}}\right)\text{ft} = \rho\Delta y$


Chain: $\int_0^{500} 2(500-y)\,dy = (-1)\int_0^{500} 2(500-y)(-1)\,dy$

$= (-1)\left[(500-y)^2\right]_0^{500} = (-1)[0 - 250,000]$

$= 250,000$ ft-lbs.

Work (Coal): (Weight)(Distance) $= 800(500) = 400,000$ ft-lbs

Total: 650,000 ft-lbs