MA 16200: Plane Analytic Geometry and Calculus II

Lecture 9: Intro to Trigonometric Integrals

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Sections Covered: 8.3

Basics

$-\ln|\cos x| = \ln|(\cos x)^{-1}|$

- $\int \sin x = -\cos x + C$
- $\int \cos x = \sin x + C$
- $\int sec^2 x \; dx = \tan x + C$
- $\int \csc x \cot x \; dx = -\csc x + C$
- $\int \sec x \tan x \; dx = \sec x + C$
- $\int \csc^2 x \; dx = -\cot x + C$
- $\int \tan x = \ln|\sec x| + C$
- $\int \csc x = -\ln|\csc x + \cot x| + C$
- $\int \sec x = \ln|\sec x + \tan x| + C$
- $\int \cot x = \ln|\sin x| + C$

$\int \frac{1}{u} =$

$\int \tan x \, dx = (-1) \int \frac{-\sin x}{\cos x} dx$

$u = \cos x$

$= -\ln|\cos x| + C_0$

$= \ln|\sec x| + C$

$\int \cot x \, dx = \int \frac{\cos x}{\sin x} dx$

$u = \sin x$

$= \ln|\sin x| + C$

$$\int \csc x \, dx = \int \csc x \left(\frac{\csc x + \cot x}{\csc x + \cot x} \right) dx$$

$$= (-1) \int \frac{-(\csc^2 x + \cot x \csc x)}{\csc x + \cot x} dx$$

$$= -\ln|\csc x + \cot x| + C$$

$$\int \sec x \, dx = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

$$= \ln|\sec x + \tan x| + C$$

Basic Idea

We want to integrate functions like:

- $\int \sin^3 \ dx$
- $\int \sin^5 x \cos^3 x \ dx$
- $\int \tan^3 x \sec x \ dx$
- $\int \csc^5 x \cot^2 \ dx$

The main idea: Use trig identities to simplify the integrand until we can use u-substitution.

Pythagorean Identity

Problem 1

Compute $\int \sin^3 x \, dx$

Pythagorean Identities:

$\sin^2 x + \cos^2 x = 1$ ☆

$1 + \cot^2 x = \csc^2 x$

$\tan^2 x + 1 = \sec^2 x$

$$\int \sin^3 x \, dx = \int \sin^2 x \sin x \, dx$$

$$= \int (1 - \cos^2 x) \sin x \, dx$$

$$= \int \sin x \, dx + \int \underbrace{\cos^2 x \sin x \, dx}_{u = \cos x}$$

$\int u^2 du$ (-1)

$$= -\cos x + \frac{1}{3}\cos^3 x + C$$

$$\int (\cos x)^2 \sin x \, dx = \int u^2 \sin x \frac{du}{-\sin x}$$

Let $u = \cos x$

$du = -\sin x \, dx$

$dx = \frac{du}{-\sin x}$

$$= -\int u^2 \, du$$

$$= -\frac{1}{3} u^3 + C_0$$

$$= -\frac{1}{3} \cos^3 x + C$$

Another Example

Problem 2

Compute $\int \cos^5 x \, dx$

$$\int \cos^4 x \cos x \, dx = \int (\cos^2 x)^2 \cos x \, dx$$

$$= \int (1 - \sin^2 x)^2 \cos x \, dx = \int (1 - 2\sin^2 x + \sin^4 x) \cos x \, dx$$

Let $u = \sin x$, $du = \cos x \, dx$

$$= \int (1 - 2u^2 + u^4) \, du = u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C_0$$

$$= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$$

Power Reduction Formulas

$(1-\cos 2x)^2$

Problem 3

Compute $\int \sin^4 x \, dx$

Power Reduction Formulas

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\int (\sin^2 x)^2 dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x) dx$$

$$= \frac{1}{4}\int dx - \frac{1}{2}\int \cos 2x \, dx + \frac{1}{4}\int \cos^2 2x \, dx$$

$$\frac{1}{4}\int \cos^2 2x \, dx = \frac{1}{4}\int \left(\frac{1+\cos 4x}{2}\right) dx = \frac{1}{4}\int \left(\frac{1}{2} + \frac{1}{2}\cos 4x\right) dx$$

$$= \int \frac{1}{8}\,dx + \frac{1}{4}\cdot\frac{1}{8}\int \cos 4x\,(4)\,dx$$

$$= \frac{1}{8}x + \frac{1}{32}\sin 4x + C_0$$

$$\int \sin^4 x = \int \frac{1}{4}\,dx - \frac{1}{2}\cdot\frac{1}{2}\int \cos 2x\,(2)\,dx + \frac{1}{4}\int \cos^2 2x\,dx$$

$$= \frac{1}{4}x - \frac{1}{4}\sin 2x + \frac{1}{8}x + \frac{1}{32}\sin 4x + C$$

$$= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

When sine has an odd power

Problem 4

Compute $\int \sin^5 x \cos^2 x \; dx$

$$\int \sin^5 x \cos^2 x \, dx = \int \sin^4 x \cos^2 x \sin x \, dx$$

$$= \int (\sin^2 x)^2 \cos^2 x \sin x \, dx = \int (1-\cos^2 x)^2 \cos^2 x \sin x \, dx$$

$$= \int (1 - 2\cos^2 x + \cos^4 x) \cos^2 x \sin x \, dx$$

$$= (-1)\int (-1)(\cos^2 x - 2\cos^4 x + \cos^6 x) \sin x \, dx$$

$u = \cos x$

$du = -\sin x \, dx$

$$= -\left[\frac{1}{3}\cos^3 x - \frac{2}{5}\cos^5 x + \frac{1}{7}\cos^7 x\right] + C$$

$$= -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C$$

When cosine has an odd power

Problem 5

Compute $\int \sin^{-\frac{3}{2}} x \cos^3 x \; dx$

$= \int \sin^{-\frac{3}{2}} x \cos^2 x \cos x \, dx = \int \sin^{-\frac{3}{2}} x \left(1 - \sin^2 x\right) \cos x \, dx$

$u = \sin x$

$du = \cos x \, dx$

$= \int \left(\sin^{-\frac{3}{2}} x - \sin^{\frac{1}{2}} x\right) \cos x \, dx$

$= -2 \sin^{-\frac{1}{2}} x - \frac{2}{3} \sin^{\frac{3}{2}} x + C$

When they both have even powers

$(\sin^2 x)^2$

Problem 6

Compute $\int \sin^4 x \cos^2 x \; dx$

$$\int \left(\frac{1-\cos 2x}{2}\right)^2 \left(\frac{1+\cos 2x}{2}\right) dx$$

$$= \frac{1}{8}\int \left(1 - \cos 2x - \cos^2 2x + \cos^3 2x\right) dx$$

$$= \frac{1}{8}\int \left(1 - \cos 2x - \frac{1}{2} - \frac{1}{2}\cos 4x + \cos^3 2x\right) dx$$

$$= \underbrace{\frac{1}{8}\int \left(\frac{1}{2} - \cos 2x - \frac{1}{2}\cos 4x\right) dx}_{(I1)} + \underbrace{\frac{1}{8}\int \cos^3 2x \, dx}_{(I_2)}$$

$(I1): \int\left(\frac{1}{16}-\frac{1}{8}\cos 2x-\frac{1}{16}\cos 4x\right)\, dx$

$= \int \frac{1}{16}\,dx - \frac{1}{2}\cdot\frac{1}{8}\int \cos 2x\,(2)\,dx - \frac{1}{4}\cdot\frac{1}{16}\int \cos 4x\,(4)\,dx$

$= \frac{1}{16}x - \frac{1}{16}\sin 2x - \frac{1}{64}\sin 4x + C_1$

$(I_2): \frac{1}{8}\int \cos^3 2x\,dx = \frac{1}{8}\int \cos^2 2x\,\cos 2x\,dx$

$= \frac{1}{8}\int (1-\sin^2 2x)\cos 2x\,dx$

$u = \sin 2x$
$du = 2\cos 2x\,dx$

$\frac{1}{8}\int \cos 2x\,dx - \frac{1}{2}\cdot\frac{1}{8}\int \sin^2 2x\,\cos 2x\,(2)\,dx$

$= \frac{1}{16}\sin 2x - \frac{1}{16}\cdot\frac{1}{3}\sin^3 2x + C_2$

$$\int \sin^4 x \cos^2 x \, dx = (I1) + (I_2)$$

$$= \frac{1}{16}x - \frac{1}{64}\sin 4x - \frac{1}{48}\sin^3 2x + C$$

General Strategy

$\int \sin^m x \cos^n x \; dx$	**Strategy**
m odd and positive, n real	Split off $\sin x$, rewrite the resulting even power of $\sin x$ in terms of $\cos x$, then use $u = \cos x$.
n odd and positive, m real	Split off $\cos x$, rewrite the resulting even power of $\cos x$ in terms of $\sin x$, then use $u = \sin x$.
m and n are both even, non-negative integers	Use half-angle formulas to transform the integrand into a polynomial of $\cos 2x$, then apply the preceding strategies once again to powers of $\cos 2x$ greater than 1.

Reduction Formula (for sine) Derivation

Problem 7

Recursively compute $\int \sin^n x \, dx$

Reduction Formulas

Theorem 8

Assume n is a positive integer:

1. $\int \sin^n x \ dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \ dx$
2. $\int \cos^n x \ dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \ dx$
3. $\int \tan^n x \ dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \ dx$, *provided* $n \neq 1$
4. $\int \sec^n x \ dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \ dx$, *provided* $n \neq 1$

Powers of Tangent

$$\tan^2 x + 1 = \sec^2 x$$

Problem 9

Compute $\int \tan^3 x \, dx$

$$\int \tan^3 x \, dx = \int \tan^2 x \tan x \, dx$$
$$= \int (\sec^2 x - 1) \tan x \, dx$$
$$= \int (\tan x \sec^2 x - \tan x) \, dx$$

$u = \tan x$

$du = \sec^2 x \, dx$

$$= \int \tan x (\sec^2 x) \, dx - \int \tan x \, dx$$
$$= \frac{1}{2} \tan^2 x - \ln|\sec x| + C$$

Factoring out $\sec^2 x$

Problem 10

Compute $\int \tan^6 x \sec^4 x \ dx$

$$\int \tan^6 x \sec^2 x \sec^2 x \, dx = \int \tan^6 x (\tan^2 x + 1) \sec^2 x \, dx$$

$$= \int (\tan^8 x + \tan^6 x) \sec^2 x \, dx$$

$u = \tan x$
$du = \sec^2 x \, dx$

$$= \frac{1}{9} \tan^9 x + \frac{1}{7} \tan^7 x + C$$

Odd power of $\sec x$ $\tan x$

Problem 11

Compute $\int \tan^5 x \sec^7 x \; dx$

$$= \int \tan^4 x \sec^6 x \; \underbrace{(\sec x \tan x)\, dx}_{du}$$

$$= \int (\sec^2 x - 1)^2 \sec^6 x \; \underbrace{[\sec x \tan x]\, dx}$$

Let $u = \sec x$; $du = \sec x \tan x \, dx$

$$= \int (u^2 - 1)^2 u^6 \, du = \int (u^4 - 2u^2 + 1) u^6 \, du$$

$$= \int (u^{10} - 2u^8 + u^6)\,du$$

$$= \frac{1}{11}u^{11} - \frac{2}{9}u^9 + \frac{1}{7}u^7 + C_0$$

$$= \frac{1}{11}\sec^{11}x - \frac{2}{9}\sec^9 x + \frac{1}{7}\sec^7 x + C$$

General Strategy

$\int \tan^m x \sec^n x \; dx$	**Strategy**
n even, m real	Split off $\sec^2 x$, rewrite the resulting even power of $\sec x$ in terms of $\tan x$, then use $u = \tan x$.
m odd and positive, n real	Split off $\sec x \tan x$, rewrite the resulting even power of $\tan x$ in terms of $\sec x$, then use $u = \sec x$.
m even and positive, n odd and positive	Rewrite the even power of $\tan x$ in terms of $\sec x$ to produce a polynomial of $\sec x$, then apply Reduction Formula 4 to each term.

Cotangent

Similar strategies can be used when the integrand is powers of cotangent and cosecant. $1+\cot^2 x = \csc^2 x$

Problem 12

Compute $\int \cot^4 x \; dx$

$$\int (\cot^2 x)^2 dx = \int (\csc^2 x - 1)\cot^2 x \, dx$$

$$= (-1)\int \cot^2 x \, (-1)\csc^2 x - \int \cot^2 x \, dx$$

$u = \cot x$

$du = -\csc^2 x \, dx$

$$= -\frac{1}{3}\cot^3 x - \int (\csc^2 x - 1) dx$$

$$= -\frac{1}{3}\cot^3 x + \int -\csc^2 x \, dx + \int dx$$

$$= -\frac{1}{3}\cot^3 x + \cot x + x \quad + C$$

Square Roots

$$\int \frac{1}{\sqrt{u}}\,du = \int u^{-\frac{1}{2}}$$

Problem 13

Compute $\int \sqrt{1-\sin x}\;dx$

$$\int \frac{\sqrt{1-\sin x}}{1}\cdot\frac{\sqrt{1+\sin x}}{\sqrt{1+\sin x}}\,dx = \int \frac{\sqrt{1-\sin^2 x}}{\sqrt{1+\sin x}}\,dx$$

$$= \int \frac{\sqrt{\cos^2 x}}{\sqrt{1+\sin x}}\,dx = \int \frac{\cos x}{\sqrt{1+\sin x}}\,dx$$

$u = 1+\sin x$

$du = \cos x\,dx$

$$= 2\sqrt{1+\sin x} + C$$

$$\sqrt{1-\cos x} = \sqrt{2\left(\frac{1-\cos x}{2}\right)}$$

$$= \sqrt{2}\cos^2\left(\frac{x}{2}\right)$$

Powers of Secant

$$\int u\,dv = uv - \int v\,du$$

"Ultra Violet Voo-doo"

Problem 14

Compute $\int \sec^3 x \; dx$

$$\int \underbrace{\sec x}_{u} \underbrace{\sec^2 x \, dx}_{dv} = \sec x \tan x - \int \tan^2 x \sec x \, dx$$

$u = \sec x \qquad v = \tan x$

$du = \sec x \tan x \, dx \qquad dv = \sec^2 x \, dx$

$$= \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx$$

$$\int \sec^3 x = \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

$$+\int \sec^3 x \, dx \qquad\qquad +\int \sec^3 x \, dx$$

$$2\int \sec^3 x \, dx = \sec x \tan x + \int \sec x \, dx$$

$$2\int \sec^3 x \, dx = \sec x \tan x + \ln|\sec x + \tan x| + C$$

$$\int \sec^3 x \, dx = \frac{1}{2}\left[\sec x \tan x + \ln|\sec x + \tan x|\right] + C$$

Product to Sum Formulas

Theorem 15

- $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

$\sin(A+B) = \sin A \cos B + \sin B \cos A$

$\sin(A-B) = \sin A \cos B - \sin B \cos A$

Using the Product to Sum Formulas

Problem 16

Compute $\int \sin \underbrace{4x}_{A} \cos \underbrace{5x}_{B} \, dx$

$$= \int \frac{1}{2}\left[\sin(-x) + \sin(9x)\right] dx$$

$$= (-1)\frac{1}{2}\int \sin(-x)(-1)\,dx + \frac{1}{9}\cdot\frac{1}{2}\int (9)\sin(9x)\,dx$$

$$= \frac{1}{2}\cos(-x) + \frac{1}{18}\left[-\cos(9x)\right]$$

$$= \frac{1}{2}\cos x - \frac{1}{18}\cos 9x + C$$