MT 2 Review

Quiz 7

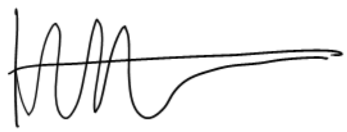
① The alternating series test does not test for absolute convergence.

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{3^n+2^n}$$

$$\sum_{n=1}^{\infty} \left|(-1)^n \frac{1}{3^n+2^n}\right| = \sum_{n=1}^{\infty} \frac{1}{3^n+2^n} \leq \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{1}{3}\right)^{n-1}$$

② If a series fails to meet the conditions of a test, you can not use the test.

$$\sum_{n=1}^{\infty} (-1)^n \underbrace{\sin\left(\frac{-\pi}{n}\right)}_{\text{Increasing Eventually}}$$



$$\sum_{n=1}^{\infty} \sin \frac{\pi}{n}$$

$$\sum_{k=1}^{\infty} \frac{k^e}{k^\pi} = \sum_{k=1}^{\infty} k^{e-\pi} = \sum_{k=1}^{\infty} \frac{1}{k^{\pi-e}}$$

p-series: $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if $p>1$, otherwise diverges $(p \leq 1)$

$3.14 - 2.78 = 0.42 < 1$

$$\sum \frac{1}{k^{-2}} \qquad \sum \frac{1}{k^2}$$

$\sum \frac{1}{k^p}$. Use integral test with $f(x) = x^{-p}$

f is continuous and positive, $f' = -px^{-p-1} < 0$ when $x \geq 1$

Integral Test: $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if and only if $\int_1^{\infty} \frac{1}{x^p}dx$ converges

$$\int_1^{\infty} x^{-p}dx \underset{\text{when } p \neq 1}{=} \left.\frac{x^{-p+1}}{-p+1}\right]_1^{\infty} = \lim_{t\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^t$$

$$= \lim_{t\to\infty}\left[\boxed{\frac{t^{-p+1}}{-p+1}} - \frac{1}{-p+1}\right]$$

$\to 0$

Case 1, when $p<1$, $t^{-p+1} \to \infty$, integral diverges

Case 2, when $p>1$, t^{-p+1}

$$\lim_{t\to\infty}\left[0 - \frac{1}{-p+1}\right] = \frac{1}{p-1}$$

Another Example
$$\int_0^{\infty} \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = \int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}dx + \int_1^{\infty} \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$$

$= \lim_{a\to 0^+} \int_a^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx + \lim_{b\to\infty} \int_1^b \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$. Remember $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$, so $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\right)dx$

$= 2e^{\sqrt{x}} + C$

$\lim_{a\to 0^+}\left[2e^{\sqrt{x}}\right]_a^1 + \lim_{b\to\infty}\left[2e^{\sqrt{x}}\right]_1^b$

$= \lim_{a\to 0^+}\left[2e - 2e^{\sqrt{a}}\right] + \underbrace{\lim_{b\to\infty}\left[2e^{\sqrt{b}} - 2e\right]}_{\text{Diverges}}$

$= (2e - 2)$

Hence the integral diverges

$\underbrace{\sum b_n}_{\text{diverges}} < \underbrace{\sum a_n}_{\text{positive terms}}$ <

$\lim \frac{a_n}{b_n} = L$; $0 < L < \infty$

Then $\sum a_n$ and $\sum b_n$ either both converge or diverge

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Growth Theorem

$\ln n \ll \underbrace{n^p}_{\text{polynomial}} \ll \underbrace{b^n}_{\text{Exponential}} \ll n! \ll n^n$

Alternating Series:

$|R_N| \leq a_{n+1}$

For taylor polynomial

$|R_N| \leq \frac{f^{(N+1)}(c)}{(N+1)!}(x-a)^{N+1}$ c is between x and a

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Test for Divergence:

$\sum a_n$ and $\lim_{n\to\infty} a_n \neq 0$, then the series diverges

$\sum_{n=1}^{\infty} \boxed{\frac{n^2}{2n^2+1}}$; $\lim_{n\to\infty} \frac{n^2}{2n^2+1} = \frac{1}{2} \neq 0$

$\sum \frac{\ln n}{n}$; $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

If $\sum a_n$ diverges, then $\sum |a_n|$ also diverges

Check $\sum |a_n|$

Geometric $\sum_{n=0} a\cdot r^n$ $\sum_{n=1} b\cdot r^{n-1}$

$\sum_{n=0}^{\infty} 2\left(\frac{1}{3}\right)^n$. Check if $\left|\frac{1}{3}\right|<1$

$\underbrace{\sum_{n=0}^{\infty} 2\left(\frac{1}{3}\right)^n}_{2+\sum_{n=1} 2\left(\frac{1}{3}\right)^{n-1}}$ $\sum_{n=0} 2\left(\frac{1}{3}\right)^n = \frac{2}{1-\frac{1}{3}} = 3$

$\sum_{n=1}^{\infty} 2\left(\frac{1}{3}\right)^n = \sum_{n=1} \frac{2}{3}\left(\frac{1}{3}\right)^{n-1}$

$= \frac{2/3}{1-1/3} = \frac{2/3}{2/3} = 1$

If $r=1$

$\sum a(1)^n$

$=\sum a$

$= a+a+a+\cdots$

$\lim_{n\to\infty} a = a$