Quiz 5: Integration Techniques, Sequences/Series (§8.5, 8.9, 10.1)

Score: /10
Name:________________________ Length: 15 minutes

Directions: Answer all the questions below in the space provided; you must show the work for full credit. Use proper notation. Clearly label the final answers.

1. Use Parts (a)-(c) to find the value of:

$$\sum_{n=1}^{\infty} \frac{6}{(3n+1)(3n+4)}$$

(a) (3 points) Decompose the summand $a_n = \frac{6}{(3n+1)(3n+4)}$ using partial fractions.

Solution:
$$\frac{6}{(3n+1)(3n+4)} = \frac{A}{3n+1} + \frac{B}{3n+4} = \frac{2}{3n+1} - \frac{2}{3n+4}$$

(b) (3 points) Use Part (a) to find an explicit formula for the sequence of partial sums $\{S_N\}_{N=1}^{\infty}$.

Solution:
$$\begin{aligned} S_1 &= \frac{2}{4} - \frac{2}{7} = \frac{1}{2} - \frac{2}{7} \\ S_2 &= \left(\frac{1}{2} - \frac{2}{7}\right) + \left(\frac{2}{7} - \frac{2}{10}\right) = \frac{1}{2} - \frac{2}{10} \\ S_3 &= \left(\frac{1}{2} - \frac{2}{10}\right) + \left(\frac{2}{10} - \frac{2}{13}\right) = \frac{1}{2} - \frac{2}{13} \\ &\vdots \\ S_N &= \frac{1}{2} - \frac{2}{3N+4} \\ &\vdots \end{aligned}$$

Thus $S_N = \frac{1}{2} - \frac{2}{3N+4};\ N \geq 1$

(c) (1 point) Compute $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{6}{(3n+1)(3n+4)}$

Solution:
$$\sum_{n=1}^{\infty} \frac{6}{(3n+1)(3n+4)} = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(\frac{1}{2} - \frac{2}{3N+4}\right) = \frac{1}{2}$$

2. Use Parts (a) and (b) to find the values of p that make the integral below converge.

$$\int_1^\infty \frac{1}{x^p}\,dx$$

(As a reminder, you must show complete work for full credit. The final answer is not enough.)

(a) (1 point) Does the integral converge or diverge when $p = 1$?

Solution:

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \ln t = \infty$$

So when $p = 1$, the integral diverges.

(b) (2 points) When $p \neq 1$, what values of p make the integral converge?

Solution: When $p \neq 1$, we integrate using the reverse power rule.

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\left[\frac{x^{1-p}}{1-p}\right]_1^t = \frac{1}{p-1} + \lim_{t\to\infty}\frac{t^{1-p}}{1-p}$$

Case 1: When $p < 1$, then $1 - p > 0$. So $t^{1-p} \to \infty$ as $t \to \infty$. Thus,

$$\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1} + \lim_{t\to\infty}\frac{t^{1-p}}{1-p} = \infty$$

Case 2: When $p > 1$, $1 - p < 0$. Thus, $t^{1-p} \to 0$ as $t \to \infty$. Computing the integral we get,

$$\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1} + \lim_{t\to\infty}\frac{t^{1-p}}{1-p} = \frac{1}{p-1} + 0 = \frac{1}{p-1} < \infty$$

Therefore, the integral converges when $p > 1$ and diverges otherwise.