Quiz 7: Comparison and Alternating Series Tests (§10.5-10.6)

Score: /10

Name:______________________________________ Length: 15 minutes

Directions: Attempt all questions; you must show work for full credit. Use proper notation. In your work, clearly label question numbers and your final answer. If you need to use another sheet of paper, make sure to write your name on it.

1. For each part, determine whether the series converges absolutely, converges conditionally, or diverges. **For full points,** show work showing why the series converges/diverges and state the test used.

   (a) (2 points) $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{3^n+2^n}$

   Solution: Recognize that $3^n + 2^n > 0$ for all $n \geq 1$. So,

   $$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{3^n + 2^n} \right| = \sum_{n=1}^{\infty} \frac{1}{3^n + 2^n} \leq \sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \frac{1}{3} \left(\frac{1}{3}\right)^{n-1}$$

   Here $\sum_{n=1}^{\infty} \frac{1}{3} \left(\frac{1}{3}\right)^{n-1}$ is a convergent geometric series, and the terms $\left\{\frac{1}{3^n+2^n}\right\}$ are positive. Therefore, the series converges absolutely by the Comparison Test.

   (b) (2 points) $\sum\limits_{n=1}^{\infty} \frac{\cos \pi n}{n^4}$

   Solution:

   $$\sum_{n=1}^{\infty} \left| \frac{\cos(\pi n)}{n^4} \right| = \sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^4} \right| = \sum_{n=1}^{\infty} \frac{1}{n^4}$$

   $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is a convergent p-series ($p = 4$). Therefore, the series converges absolutely by the p-series test.

   (c) (2 points) $\sum\limits_{n=1}^{\infty} (-1)^n \frac{n(n^4+1)}{2n^5}$

   Solution: The non-oscillating part goes to $\frac{1}{2}$ as $n \to \infty$. So it will diverge by the Test for Divergence.
   **NOTE: The starting index was technically 0 (which makes the first term undefined). That was a typo, but you could technically argue divergence because of that.

   (d) (4 points) $\sum\limits_{n=3}^{\infty} (-1)^{n+1} \frac{n-1}{n^2+3}$

   Solution: Test for absolute convergence by the limit comparison test:

   $$\lim_{n\to\infty} \frac{\frac{n-1}{n^2+3}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2 - n}{n^2 + 3} = 1$$

$0 < 1 < \infty$ and $\sum \frac{1}{n}$ diverges. So, $\sum_{n=3}^{\infty} \left|(-1)^{n+1}\frac{n-1}{n^2+3}\right| = \sum_{n=3}^{\infty} \frac{n-1}{n^2+3}$ diverges. However, $\frac{n-1}{n^2+3}$ is decreasing on $(3, \infty)$. To see, consider $f(x) = \frac{x-1}{x^2+3}$:

$$f'(x) = \frac{x^2+3-2x(x-1)}{(x^2+3)^2} \overset{set}{=} 0$$

Solving $x^2 - 2x - 3 = 0$ f has a critical point at $x = 3$. $f'(1000000) < 0$, so f is decreasing on $(3, \infty)$. Also,

$$\frac{n-1}{n^2+3} \to 0 \text{ as } n \to \infty$$

Therefore, the series converges conditionally by the Alternating Series Test