Quiz 8: Ratio/Root Test, Taylor Polynomials (§10.7, 10.8, 11.1)

Score: /10

Name:____________________________________

Length: 15 minutes

Directions: Attempt all questions; you must show work for full credit. Use proper notation. In your work, clearly label question numbers and your final answer. If you need to use another sheet of paper, make sure to write your name on it.

1. For the series $\sum_{n=1}^{\infty} a_n$, suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$. Use this information to answer each part below (**No work is needed, the final answer is enough**).

   (a) (1 point) According to the Ratio Test, what values of L imply the series $\sum_{n=1}^{\infty} a_n$ **converges (absolutely)**?

   Solution: The series converges if $L < 1$

   (b) (1 point) According to the Ratio Test, what values of L imply the series $\sum_{n=1}^{\infty} a_n$ **diverges**?

   Solution: The series diverges if $L > 1$

   (c) (1 point) What is the conclusion of the Ratio Test if $L = 1$?

   Solution: The test is inconclusive.

   (d) (1 point) What is the conclusion of the Ratio Test if $L = \infty$?

   Solution: The series diverges if $L = \infty$.

2. (3 points) Determine if the series below converges or diverges. **For full points**, explain why the series converges/diverges and state the test used.
   [Hint: Remember $\lim_{n\to\infty}(1+\frac{x}{n})^n = e^x$ and $e^x \geq 1+x$ for any x].

$$\sum_{n=1}^{\infty} \left(1+\frac{3}{n}\right)^{n^2}$$

Solution:

$$\lim_{n\to\infty} \left(\left|\left(1+\frac{3}{n}\right)^{n^2}\right|\right)^{1/n} = \lim_{n\to\infty} \left(1+\frac{3}{n}\right)^{n} = e^3 \geq 1+3 = 4 > 1$$

The series diverges by the Root Test.

3. Answer each part below:

(a) ($1\frac{1}{2}$ points) Find the linear approximation of e^x at $x = 0$.

Solution: Let $f(x) = e^x$. We approximate using the 1st order Taylor Polynomial p_1 at $x = 0$. Near $x = 0$,

$$e^x \approx f(0) + f'(0)(x-0) = e^0 + e^0 x = 1 + x$$

(b) (1 point) Find the quadratic approximation of e^x at $x = 0$.

Solution: Let $f(x) = e^x$. We approximate using the 2nd order Taylor Polynomial p_2 at $x = 0$. Near $x = 0$,

$$e^x \approx f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 = e^0 + e^0 x + \frac{e^0}{2}x^2 = 1 + x + \frac{1}{2}x^2$$

(c) ($\frac{1}{2}$ point) One can bound the error $|R_N|$ of using the N-th Taylor Polynomial to approximate e^x by:

$$0 \leq |R_N| \leq e^{|x|}\frac{|x|^{N+1}}{(N+1)!}$$

Does the error go to 0 as $N \to \infty$? [Hint: $\lim\limits_{N\to\infty} |R_N| = 0$ if $\lim\limits_{N\to\infty} \frac{|x|^{N+1}}{(N+1)!} = 0$ for a fixed x.]

(**No work is needed, the answer is enough**).

Solution: Yes, the error decays to 0 as $N \to \infty$. Since $\lim_{N\to\infty} e^{|x|}\frac{|x|^{N+1}}{(N+1)!} = 0$ for a fixed x, one has $\lim_{N\to\infty} |R_N| = 0$ by the Squeeze Theorem.