Quiz 9: Power Series (§11.2, 11.3)

Score: /10

Name:____________________________________ Length: 15 minutes

Directions: Attempt all questions; you must show work for full credit. Use proper notation. In your work, clearly label question numbers and your final answer. If you need to use another sheet of paper, make sure to write your name on it.

1. (4 points) Determine the radius and interval of convergence for the power series:

$$\sum_{n=1}^{\infty} \frac{(2x)^n}{\sqrt{n}}$$

Solution: From the Ratio Test,

$$\lim_{n\to\infty} \left| \frac{(2x)^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(2x)^n} \right| = |2x| \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = |2x| < 1$$

Therefore, $|x| < \frac{1}{2}$ and the radius of convergence is $\frac{1}{2}$. When $x = \frac{1}{2}$, it is a divergent p-series ($p = \frac{1}{2}$). When $x = -\frac{1}{2}$, the series converges by the Alternating Series Test.

Therefore, the interval of convergence is $[-\frac{1}{2}, \frac{1}{2})$.

2. The power series representation of e^x is given below:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}; \quad x \in (-\infty, \infty)$$

(a) (2 points) Find a power series representation for e^{-x^2} centered at 0. The interval of convergence is $(-\infty, \infty)$, but you do **NOT** need to show this.

Solution:

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$$

(b) (2 points) Use Part (a) to find an anti-derivative for e^{-x^2}. Express your final answer as a power series.

Solution:

$$\int e^{-x^2}\, dx = \int \sum_{n=0} \frac{(-1)^n}{n!} x^{2n}\, dx \overset{x\in(-\infty,\infty)}{=} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int x^{2n}\, dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\cdot n!} x^{2n+1}$$

where C is the constant of integration.

3. (2 points) The Maclaurin Series for $\frac{1}{1-x}$ is given below:

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n; \quad x \in (-1, 1)$$

What is the value of $f^{(100)}(0)$? [**Hint:** What is the $n = 100$th term in the Maclaurin Series?]

Solution: Since it is a Maclaurin Series, on $(-1, 1)$:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

By comparing coefficients, $\frac{f^{(n)}(0)}{n!} = 1$ for any n. When $n = 100$,

$$\frac{f^{(100)}(0)}{100!} = 1$$
$$f^{(100)}(0) = 100!$$