Test 1: Applications of Integration/Integration Techniques

Volume of Solids of Revolution

1. Disc Method (Ideal for a when single function revolves around x-axis)

Revolved Around x-axis	Revolved Around y-axis
$A(x) = \pi \int_a^b f(x)^2 dx$	$A(y) = \pi \int_c^d f(y)^2 \, \mathrm{d}y$

2. Washer Method (Ideal for when the area in between functions is revolved around either axis)

Revolved Around x-axis	Revolved Around y-axis
$A(x) = \pi \int_a^b \left(f(x)^2 - g(x)^2\right) \mathrm{d}x$ $where\ f(x) > g(x)\ \forall x\ on\ (a, b)$	$A(y) = \pi \int_c^d \left(f(y)^2 - g(y)^2\right) \mathrm{d}y$ $where\ f(y) > g(y)\ \forall y\ on\ (c, d)$

3. Shell Method (Ideal for when one function is revolved around y-axis and/or is nearly impossible to put in terms of y)

Revolved Around x-axis	Revolved Around y-axis
$A(y) = 2\pi \int_c^d y f(y) \, \mathrm{d}y$	$A(x) = 2\pi \int_a^b x f(x) \, \mathrm{d}x$

Work:

- Constant Force and Finite Displacement
  $\mathrm{W} = \mathrm{Fd} \Rightarrow \mathrm{Work} = \mathrm{Force} \cdot \mathrm{Displacement}$
- Variable Force over an interval of displacement
  $$W = \int_a^b F(x) \, \mathrm{d}x$$
  $\mathrm{a} = \text{intital displacement},\ \mathrm{b} = \text{ final displacement},\ \mathrm{F}(\mathrm{x}) = \text{Force Function in terms of Displacement}$
- Constant Force and Variable Displacement over an interval
  $$W = F \int_a^b s(x) \, \mathrm{d}x;\ s(\mathrm{x}) = \text{Displacement Function}$$
  - Units for Work
    - Imperial: foot-pound(s) [ft-lbs.]
    - Metric/SI: Joule(s) [J] = Newton-Meters [Nm] = $\mathrm{kg}\frac{\mathrm{m}^2}{\mathrm{s}^2}$

Integration Techniques:

- **Memorize the following list of integrals:**

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C$$

$$\int \csc x \cot x\,dx = -\csc x + C$$

$$\int \sinh x\,dx = \cosh x + C$$

$$\int \cosh x\,dx = \sinh x + C$$

$$\int \tan x\,dx = \ln|\sec x| + C$$

$$\int \cot x\,dx = \ln|\sin x| + C$$

$$\int \frac{1}{x^2 + a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

- U-Substitution for Indefinite Integrals (From Calculus I)
  - Generally used to undo the Chain Rule/Integrating Compositions
  - Assign the inner function to a variable (typically u)
  - Then, find du as a differential. Solve explicitly for the other differential (usually dx)
  - If u=g(x) and du= g'(x)dx (OR dx=du/g'(x)). Then,
    $$\int f(g(x))g'(x)dx = \int f(u)du$$
  - Integrate in terms of du
  - Replace u with what you replaced it with

- U-Substitution for Definite Integrals (From Calculus I)
  - Same process, except express the integral completely in terms of u

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

- Informal U-Substitution, "The Reverse Chain Rule"

  $$\int f(g(x)) \cdot g'(x)\,dx = F(g(x)) + C$$

- Integration by Parts
  - Use to undo the Product Rule/When a polynomial is multiplied by a Transcendental
    - Formula in terms of f(x) and g(x)

      $$\int f(x)\,g'(x)\,\mathrm{dx} = f(x)\,g(x) - \int f'(x)\,g(x)\,\mathrm{dx}$$

    - Formula with the given Substitution

      $$Let: \begin{cases} u = f(x) & du = f'(x)\,dx \\ v = g(x) & dv = g'(x)\,dx \end{cases} \Rightarrow \int udv = uv - \int vdu$$

- Integration Trigonometric Functions (General Strategies)
  - See 7.2 for more details
    - The following functions are easier to integrate then they are paired with the other, put them in terms of these pairs (or when in doubt, in terms of sine and cosine)
      - Sine and Cosine
      - Tangent and Secant
      - Cosecant and Cotangent
    - Use trig identities to simplify the integral in order to use U-Substitution
      - The Identities used will vary based on the degrees of the function (whether it is raised by an odd or even power, compositions, etc.)
  - **Learn the following Identities:**
    - Pythagorean Identities

      $$\sin^2(\theta) + \cos^2(\theta) = 1$$

      $$\tan^2(\theta) + 1 = \sec^2(\theta)$$

      $$1 + \cot^2(\theta) = \csc^2(\theta)$$

    - Double Angle Formulas (For Sine and Cosine)

      $$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

      $$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)$$

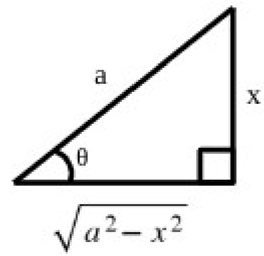
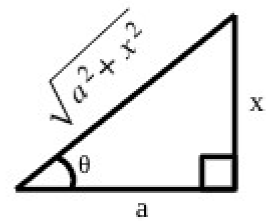
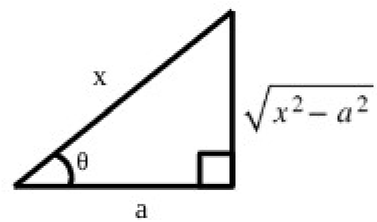
    - Power Reduction formulas (For Sine and Cosine)

      $$\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))$$

      $$\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$$

- Trigonometric Substitution
  - Here is a table showing the appropriate substitutions
  - <u>Note: Knowing the derivatives of inverse sine, inverse tangent, and inverse secant can help in memorizing these</u> :

Function	Substitution	Domain of Substitution	Triangle Representation

$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$	

- Partial Fraction Decomposition
  - Used for integrating rational expressions
    - If it is a improper fraction (the highest degree of the polynomial in the numerator is larger than that of the denominator), then divide
    - Factor the denominator
    - Decompose the fraction
      - Linear Terms:
        $$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1 x + b_1} + \frac{A_2}{a_2 x + b_2} + \cdots + \frac{A_k}{a_k x + b_k}$$
        - k= the power the factor is raised to
          - Repeat for each linear factor
      - Quadratic Terms:
        $$\frac{A_1 x + B_1}{ax^2 + bx + c} + \frac{A_2 x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_r x + B_r}{(ax^2 + bx + c)^r}$$
        - r= the power the factor is raised to
          - Repeat for each quadratic term
    - Find the Coefficients (A_n or different letters are often used)
      - Multiply by Q(x) on both sides
      - Group like terms and factor the " x^n " part of terms
      - Compare the coefficients of R(x), they should be equal
        - Constants are equal, x term's coefficients is equal, x^2 term's coefficients are equal etc.
    - Plug the coefficients into the partial fractions and integrate

Improper Integrals

- **Improper Integral:** When either one (or both) of the bounds are infinity, or there is a vertical asymptote on the interval being integrated
- General rule: Assign the infinity/asymptote part to a variable and take the limit as x approaches that value
  - Infinite Bounds of Integration:
    - Positive Infinity

      $$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx = \lim_{t\to\infty} F(x)\,\big]_a^t$$

    - Negative Infinity

      $$\int_{-\infty}^{b} f(x)\,\mathrm{d}x = \lim_{t\to-\infty}\int_t^b f(x)\,\mathrm{d}x = \lim_{t\to-\infty} F(x)\,\big]_t^b$$

    - Both bounds are infinity
      - Split it at an x value (typically 0) to create one negative infinity integral and one positive integral
      - If one integral diverges (defined later in guide), then the whole integral diverges
  - Integrating on (a,b) with vertical asymptote at x=c

    $$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x = \lim_{t\to c^-}\int_a^t f(x)\,\mathrm{d}x + \lim_{s\to c^+}\int_s^b f(x)\,\mathrm{d}x$$

    - If one integral diverges, then the whole integral diverges

Test 2: Applications of Integration, Elementary Differential Equations, Parametric Equations, Polar Coordinates, Series/Sequences Intro:

Arc Length:

- The arc length of f(x) on [a,b] is defined as:

  $$L = \int_a^b \sqrt{1+[f'(x)]^2}\,\mathrm{d}x \Leftrightarrow L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,\mathrm{d}x$$

- The arc length of f(y) on [c,d] is defined as:

  $$L = \int_c^d \sqrt{1+[f'(y)]^2}\,\mathrm{d}y \Leftrightarrow L = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,\mathrm{d}y$$

Surface Area of Solids of Revolution

	About the x-axis
Newton's Notation	$S = \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$
Leibniz's Notation	$S = \int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

Separable Equations:

- Written (or can be written) as:

  $$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

  - Cross Multiply and Integrate

$$f(x)\,dx = g(y)\,dy \Rightarrow \int f(x)\,dx = \int g(y)\,dy \Rightarrow F(x) + C_1 = G(y) + C_2 \Rightarrow G(y) + C;\ C = C_1 - C_2$$

- Solve for y
- When given an initial conditions
  - Solve for y, then (when given f(x)=y), plug in the value of x and y and solve for the constant

Exponential Growth/Decay

- Written in the form:

  $$A = P_o e^{kt}$$

  - A = Dependent Value
  - P_o = initial sample size
  - e = natural growth constant
  - k = growth/decay rate
    - If k is positive, then it is exponential growth
    - If k is negative, then it is exponential decay
  - t = independent variable (usually time)
- Generally a good idea to solve for k

Parametric Equations:

- A relation whose coordinates are defined as functions of another variable
- Let the x coordinate be defined as a function with an independent variable *t*
  - x = f(t)
- Let the y coordinate be defined as another function wit an independent variable *t*
  - y = g(t)
- Coordinates of a parametric curve are (f(t) , g(t))
- Graphing parametric curves
  - Plug in values of t to get coordinates
  - Graph the following curve
  - Use arrows to indicate the "flow" of t (as t increase, what direction does the curve go)
- Solving a parametric curve
  - Use substitution to eliminate the parameter *t*

Calculus with Parametric Curves

- Derivative of a Parametric curve

  $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

- Second Derivative of a Parametric Curve

  $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

- Arch Length for a Parametric Curve

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

- Surface Area for a Parametric Curve

$$S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Polar Coordinates

- Has coordinate (r, θ) where r is the radius (distance from the origin) and θ is the angle of the arc created
- A polar curve function r = f(θ) is a fancy parametric curve
  $x = r \cdot \cos\theta \quad y = r \cdot \sin\theta$
- Equations to convert Cartesian and Polar Coordinates (including the two above)

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$$

Calculus with Polar Curves

- Derivative of a polar curve (modified derivative of a parametric curve)

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

- Area created by a polar curve

$$A = \frac{1}{2}\int_{a}^{b} r^2 d\theta$$

- Arc Length of a Polar Curve

$$L = \int_{a}^{b} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Test 3: Sequences, Series, Elementary Linear Algebra:

Sequences:

- **Sequence:** A list of numbers in a defined order
  - Notation:
  - $\{a_1, a_2, a_3, ...\} = \{a_n\}$ or $\{a_n\}^{\infty}_{n=1}$
- **Convergence v. Divergence**
  If $\lim_{n \to \infty} a_n = L$ where $L \in \mathbb{R}$, the sequence **converges** to L

  Otherwise, the sequence **diverges** (or is **divergent**)
- Note: Limit Rules apply to sequences
- The sequence $\{r^n\}$ converges if $-1 < r \leq 1$, converges to 0 if $|r| < 1$ and converges to 1 if $r=1$
- Defining Increasing, Decreasing, and Monotonic
  - The sequence $\{a_n\}$ is **increasing** if $a_n < a_{n+1}$ $\forall n \geq 1$
  - The sequence $\{a_n\}$ is **decreasing** if $a_n > a_{n+1}$ $\forall n \geq 1$
  - If it is strictly increasing or decreasing, then the sequence is **monotonic**

- Bounded Sequences:
    - A sequence $\{a_n\}$ is **bounded above** if there is a number M such as $a_n \leq M \ \forall n \geq 1$
    - A sequence $\{a_n\}$ is **bounded below** if there is a number *m* such as $m \leq a_n \ \forall n \geq 1$
    - If $\{a_n\}$ is bounded above or below, then $\{a_n\}$ is a **bounded sequence**
- Monotonic Sequence Theorem
    - THM: Every bounded, monotonic sequence is convergent

Series:
- Definitions
    - When the terms of a sequences are added, it is called a **series**
    - When the terms are added forever, it is called an **infinite series**
        - Notation: $\sum_{n=1}^{\infty} a_n$
    - The harmonic series is defined as: $\sum_{n=1}^{\infty} \frac{1}{n}$
    - The harmonic series is divergent

TABLE 9.2 Summary of Tests for Series

Test	*Series*	*Converges*	*Diverges*	*Comment*
nth-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series	$\sum_{n=0}^{\infty} ar^n$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	Sum: $S = \frac{a}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+1})$	$\lim_{n\to\infty} b_n = L$		Sum: $S = b_1 - L$
p-Series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$p \leq 1$	
Alternating Series	$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$	$0 < a_{n+1} \leq a_n$ and $\lim_{n\to\infty} a_n = 0$		Remainder: $\lvert R_N\rvert \leq a_{N+1}$
Integral (f is continuous, positive, and decreasing)	$\sum_{n=1}^{\infty} a_n,\ a_n = f(n) \geq 0$	$\int_1^{\infty} f(x)\,dx$ converges	$\int_1^{\infty} f(x)\,dx$ diverges	Remainder: $0 < R_N < \int_N^{\infty} f(x)\,dx$
Root	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} < 1$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} > 1$	Test is inconclusive if $\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} = 1$.
Ratio	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert < 1$	$\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert > 1$	Test is inconclusive if $\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert = 1$.
Direct Comparison $(a_n, b_n > 0)$	$\sum_{n=1}^{\infty} a_n$	$0 \leq a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges	$0 \leq b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges	
Limit Comparison $(a_n, b_n > 0)$	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ diverges	

Power Series:

- In the form:

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

  - “The powers series centered a x=a with coefficients c_0, c_1, c_2 etc”
- Converges to the corresponding function
- 3 possible ways the series can converge
  - Converges only at x=a
  - Converges $\forall x$
  - Converges when $|x-a|<R$ where R is called the **radius of convergence**
- Use the ratio test to determine radius of convergence
- Use another test to test the endpoints
  - Plug one of the endpoints into the series to see if that series is convergent or divergent
  - Test the other endpoint
  - Use interval notation to express the **interval of convergence**

Expressing functions as power series

- A function written in the form $\frac{a}{1-r}$ can be expressed as the geometric series $\sum_{n=0}^{\infty} a \cdot r^n$
- **Term-By-Term Differentiation/Integration**
  - Taking the Derivative of a Power Series

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

  - Integrating a Power Series

$$\int \sum_{n=1}^{\infty} c_n(x-a)^n\,dx = C + \sum_{n=1}^{\infty} \frac{c_n(x-a)^{n+1}}{n+1}$$

    - The radius of convergence of the new series is the same as the original

Taylor and Maclaurin Series

- The <u>taylor series</u> around x=a is used for approximating the function around a and it written as:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

- When a taylor series is centered at x=0, it is called the <u>maclaurin series</u> for f(x) and is defined as:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

- Maclaurin Series for Common Functions

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad (-1, 1)$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad (-\infty, \infty)$$

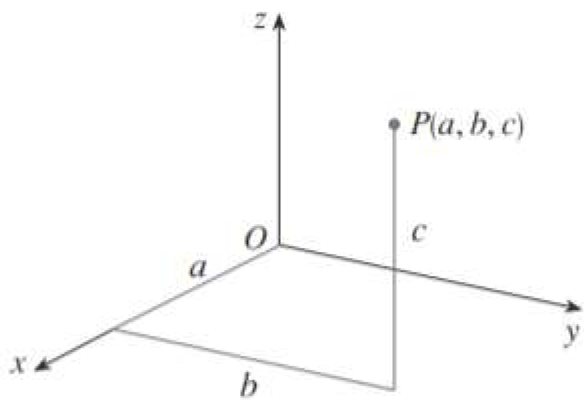
$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad (-\infty, \infty)$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad (-\infty, \infty)$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad [-1, 1]$$

3-Dimensional Coordinate Systems

- A point in $\mathbb{R}^3$ is represented as (a,b,c)



- Distance between points $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ in $\mathbb{R}^3$

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- Equation for a Sphere with a center point (h,k,l) and a radius r

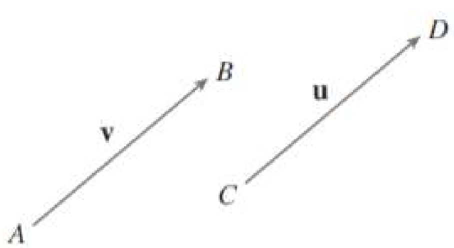
$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

- Equation for a Sphere centered at the origin

$$x^2 + y^2 + z^2 = r^2$$

Vectors

- **Vector:** A quantity with both magnitude and direction



- Notation:
  - Handwritten

    $$\overrightarrow{AB} = \vec{v} = \hat{v}$$
  - Computer/Digitally
    - Bold face (Vector v = **v**)
- Point A of **v** is called the **initial point** (or tail) and Point B is called the **terminal point** (or head)
- Since **u** and **v** are equal in magnitude and direction (despite being in seperate locations), **u**=**v**
- The zero vector, denoted **0**, has 0 magnitude and no specific direction
- Vector Addition (also called the Triangle Law)

- If **u** and **v** are vectors positioned so the initial point of **v** is at the terminal point of **u**, then **u** + **v** is the vector from the initial point of **u** to the terminal point of **v**
- Parallelogram Law: Vector addition is commutative (**u** + **v** = **v** + **u**)

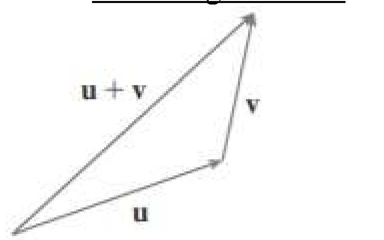


FIGURE 3 The Triangle Law

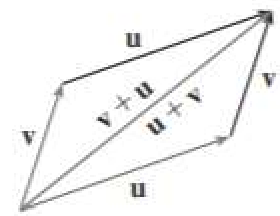


FIGURE 4 The Parallelogram Law

- Scalar Multiplication
  - If c is a scalar and **v** is a vector, then c**v** is the vector whose length is |c| times the length of **v** and whose direction is the same as **v** if c>0 and is opposite to **v** if c<0. If c= 0 or **v**=**0**, then c**v**=**0**
- Vector Subtraction
  - Add the vector with the opposite direction of the second vector
    - **u** - **v** = **u** + (-**v**)
- Each vector has <u>components</u> which show how far along the vector travels along each axis
  - **a** = <1,2> ; 1 unit along x-axis and 2 units along y-axis
- Components of a vector **a** with an initial point A(x_1,y_1,z_1) and terminal point B(x_2,y_2,z_2)

  $$\boldsymbol{a} = \left\langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \right\rangle$$
- The **magnitude** of **a** is the length of the vector and as denoted as |**a**| or ||**a**||
  - Magnitude of a 2D Vector **a**=<a_1,a_2>

    $$\|\boldsymbol{a}\| = \sqrt{a_1^2 + a_2^2}$$
  - Magnitude of a 3D Vector **a**=<a_1,a_2,a_3>

    $$\|\boldsymbol{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$
- Vector addition/subtraction for a vector in component form is the vector that is created when the corresponding components are added/subtracted together
- Scalar multiplication for a vector in component form is the vector that is created when the scalar is applied to each component
- Any vector can be expressed as a *linear combination* of the basis vectors **i**=<1,0,0>, **j**=<0,1,0>, and **k**=<0,0,1>
  - **a**=<a_1,a_2,a_3**>** ⇒ **a** = a_1**i** + a_2**j** + a_3**k**
- The unit vector (**u**) or **a** is defined as:

  $$\boldsymbol{u} = \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|}$$

Dot Product (Also called scalar product or inner product):

- If **a**=<a_1,a_2,a_3> and **b**=<b_1,b_2,b_3>, then the dot product (denoted as **a** ・ **b**) is defined as:

  $$\boldsymbol{a} \cdot \boldsymbol{b} = \left\langle a_1 b_1 + a_2 b_2 + a_3 b_3 \right\rangle$$
- Physics definition of the dot product
  - Let θ be the angle in between **a** and **b**, **a** ・ **b** is defined as:

    $$\boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\| \, \|\boldsymbol{b}\| \cos\theta$$
- Corollary to Physics Definition

$$\boldsymbol{a}\cdot\boldsymbol{b}=\|\boldsymbol{a}\|\,\|\boldsymbol{b}\|\cos\theta \Leftrightarrow \cos\theta=\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|}$$

- Vectors **a** and **b** are <u>orthogonal</u> if the angle in between them is $\pi/2$
- Determining if Vectors are parallel or orthogonal
  - Vectors **a** and **b** are orthogonal iff $\mathbf{a}\cdot\mathbf{b}=0 \Rightarrow \cos\theta=0$
  - Vectors **a** and **b** are parallel and facing the same direction iff $\mathbf{a}\cdot\mathbf{b}=\|\mathbf{a}\|\,\|\mathbf{b}\| \Rightarrow \cos\theta=1 \Rightarrow \theta=0$
  - Vectors **a** and **b** are parallel but facing the opposite direction iff $\mathbf{a}\cdot\mathbf{b}=-\|\mathbf{a}\|\,\|\mathbf{b}\| \Rightarrow \cos\theta=-1 \Rightarrow \theta=\pi$
- The **direction angles** of a nonzero vector **a** are the angles α, β, and γ (in the interval $[0,\pi]$) that **a** makes with the positive x-, y-, and z-axes
  - Equations dealing with direction angles

$$\cos\alpha=\frac{\mathbf{a}\cdot i}{\|\mathbf{a}\|\,\|\mathbf{i}\|}=\frac{a_1}{\|a\|}\qquad \cos\beta=\frac{\mathbf{a}\cdot j}{\|\mathbf{a}\|\,\|j\|}=\frac{a_2}{\|a\|}\qquad \cos\gamma=\frac{\mathbf{a}\cdot k}{\|\mathbf{a}\|\,\|k\|}=\frac{a_3}{\|a\|}$$

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$$

$$\frac{\mathbf{a}}{\|\mathbf{a}\|}=\langle\cos\alpha,\cos\beta,\cos\gamma\rangle$$

- Projections:
  - The scalar projection (also called the component) of **b** onto **a** (denoted as $\text{comp}_{\mathbf{a}}\mathbf{b}$) is defined as:

$$comp_{\,a}\boldsymbol{b}=\|\boldsymbol{b}\|\cos\theta=\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\|\boldsymbol{a}\|}$$

  - The vector projection of **b** onto **a** (denoted as $\text{proj}_{\mathbf{a}}\mathbf{b}$) is defined as:

$$proj_{\,a}\boldsymbol{b}=\left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\|\boldsymbol{a}\|}\right)\frac{\mathbf{a}}{\|\mathbf{a}\|}=\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$$

- Work problems involving projects
  - Work is force multiplied by the displacement, one needs to know the magnitude of the force that is applied in the same direction as the net displacement. So, work is displacement times the scalar projection of the force vector onto the displacement vector.

$$W=\|\boldsymbol{D}\|\ comp_{\,\boldsymbol{D}}\boldsymbol{F}$$

  - Work is measured in Joules (J) in the metric/SI system, and foot-pounds (ft-lbs) in the Imperial System

Cross Product (Also called vector product):

- Review of Determinants
  - For a 2 by 2 Matrix

$$\text{Let } \mathrm{A}=\begin{bmatrix} \mathrm{a} & \mathrm{b}\\ \mathrm{c} & \mathrm{d}\end{bmatrix},\ \text{then } \det(\mathrm{A})=\begin{vmatrix} \mathrm{a} & \mathrm{b}\\ \mathrm{c} & \mathrm{d}\end{vmatrix}=\mathrm{ad}-\mathrm{bc}$$

  - For a 3 by 3 Matrix
    - Pick a row/column
    - Know the sign for each corresponding element (shown to the right)
    - Multiply the first number of the row/column that was selected by the appropriate sign as well as the determinant of the 2 by 2 matrix

$$\begin{bmatrix} + & - & +\\ - & + & -\\ + & - & +\end{bmatrix}$$

that is created when the row and column that the first number is in has been crossed out
        - Repeat for each number in the row/column, then add the values together
        - In the example below, the first row is used

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

- The cross product for **a**=<a_1,a_2,a_3> and **b**=<b_1,b_2,b_3> (denoted **a** x **b**) is defined as:
    - Note: **a x b, i, j, and k** should be bolded, it did not come out well

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k$$

- Footnotes to definition
    - The cross product is a vector while the dot product is a number
    - The cross product is only defined for 3D vectors
    - Do not interchange rows, **a** x **b** ≠ **b** x **a**
- Thm: The vector **a** x **b** is orthogonal to both **a** and **b** since the dot product of **a** x **b** with **a** or **b** =0
- Physics definition of cross product
    - If θ is the angle between **a** and **b** (so 0≤ θ≤ π) then,

$$\|a \times b\| = \|a\| \, \|b\| \sin\theta$$

- Corollary to physics definition
    - Two nonzero vectors **a** and **b** are parallel iff **a** x **b**= **0**
- Scalar Triple Product Formula

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

- Area of Volume using cross products
    - ||**a** x **b**|| is equal to the area of the parallelogram determined by **a** and **b**
    - The volume of the parallelepiped determined by the vectors **a**, **b**, and **c** is the absolute value of the triple scalar product
        - V= |a · (**b** x **c**)|
            - If V=0, then **a**, **b**, and **c** are <u>coplanar</u>
- Torque Problems
    - Torque is defined as the force required to rotate an object about an axis. Torque (denoted by the greek letter tau (**τ**)) is the cross product between the force vector (**F**) and the lever arm vector ®.
        - The lever is defined as the radius of the circle that would be created when the end is rotated about the aforementioned axis.

$$\|\tau\| = \|\boldsymbol{r} \times \boldsymbol{F}\| = \|\boldsymbol{r}\| \, \|\boldsymbol{F}\| \sin\theta$$

- Units for Torque is Newton-Meters (Nm)
- Positive torque is a counterclockwise rotation, a negative torque is a clockwise rotation