MA 261 ONLINE, SUMMER 2026
MIDTERM EXAM 2

Name (Print): KEY ______________________ PUID: ______________________

Section Number: ______________________ Name of TA: ______________________

Start Time: ______________________ End Time: ______________________

Directions:

- Please use a **pen** if possible. Marks made by pen scan better than marks made by pencil.
- This is a **one-hour** exam.
- There are **12 questions** worth 8 points each; the maximum score is 100 points.
- The remaining 4 points are earned for filling out the information at the top of the page correctly. Section numbers, and their associated TA, are provided in the table below.

900	Eric Javier Pabon Cancel	**905**	Harrison Osprey Lemley
901	Jose Manuel Barrientos Lopez	**906**	Tyler Wallace Bruhnke
902	Leo Shen	**907**	Giovani Chuc Thai
903	Nick Polychronidis	**908**	Benjamin Scott Marlin
904	Matthew Charles Wackerfuss	**909**	Risa Michelle Fines

- Circle one and only one choice for each multiple-choice problem (Problems 1-3). **No partial credit** will be given on multiple-choice problems.
- Show all relevant work on each free-response problem (Problems 4-12). You must include some work or justification to receive full credit on these problems; partial credit may be given for steps leading to the correct solution. **Write the final answer in the box(es) provided**.
- Any external help or communication is not allowed, except with the proctor for logistical reasons. This exam is closed book and closed notes. **Calculators are not allowed**. Electronic devices, including smart watches, are to be turned off and put away.
- The last sheet is scratch paper. You may use only the scratch paper provided in this exam booklet. No additional scratch paper is permitted.

Purdue University faculty and students commit themselves towards maintaining a culture of academic integrity and honesty. The students taking this exam are not allowed to seek or obtain any kind of help from anyone to answer questions on this test. If you have questions, consult only an instructor or a proctor. You are not allowed to look at the exam of another student. You may not compare answers with anyone else or consult another student until after you finish your exam and hand it in to a proctor or to an instructor. You may not consult notes, books, calculators, cameras, or any kind of communications devices until after you finish your exam and hand it in to a proctor or to an instructor. If you violate these instructions you will have committed an act of academic dishonesty. Penalties for academic dishonesty can be very severe and may include an F in the course. All cases of academic dishonesty will be reported to the Office of the Dean of Students. Your instructor and proctors will do everything they can to stop and prevent academic dishonesty during this exam. If you see someone breaking these rules during the exam, please report it to the proctor or to your instructor immediately. Reports after the fact are not very helpful.

I agree to abide by the instructions above:

Student Signature: ______________________

Proctor Signature: ______________________

Question	Points	Score
1	8	
2	8	
3	8	
4	8	
5	8	
6	8	
7	8	
8	8	
9	8	
10	8	
11	8	
12	8	
Correct Header Info	4	
Total:	100	

1. [8 points] Let M and m be the maximum and minimum of $f(x, y) = x^2 + 2y^2$ on the closed unit disk $D = \{(x, y) |\ x^2 + y^2 \leq 1\}$ respectively. What is the value of $M + m$?

   A. -1 [Did the problem for $x^2 - 2y^2$]

   B. 0 [only checked interior]

   C. 1 [when checking bdy, forgot end points]

   D. 2

   E. 3 [assumed constraint was $x^2 + y^2 = 1$]

Interior: $f(x,y) = x^2 + 2y^2$

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} 2x = 0 \\ 4y = 0 \end{cases} \Rightarrow (0,0) \text{ is a critical point}$$

Boundary $(x^2 + y^2 = 1)$:

$$f(x,y) = x^2 + 2y^2$$

$$f(y) = (1 - y^2) + 2y^2 \ ; \ y \in [-1, 1]$$

$$= 1 + y^2$$

$$f'(y) = 2y \overset{\text{set}}{=} 0$$

Check f Values:

$f(0,0) = 0 \leftarrow$ Min

y	-1	0	1
	2	1	2

$\uparrow$ Max $\Rightarrow M + m = 2 + 0 = 2$

2. [8 points] Rewrite the integral:

$$\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} f(x,y,z)\ dzdydx$$

as an iterated integral in the order $dxdydz$.

A. $\int\limits_{0}^{1-y}\int\limits_{x^2}^{1}\int\limits_{-1}^{1} f(x,y,z)\ dxdydz$ [didn't change bounds]

B. $\int\limits_{0}^{1}\int\limits_{0}^{1-z}\int\limits_{0}^{\sqrt{y}} f(x,y,z)\ dxdydz$ [drop negative branch of x = +/- sqrt(y)

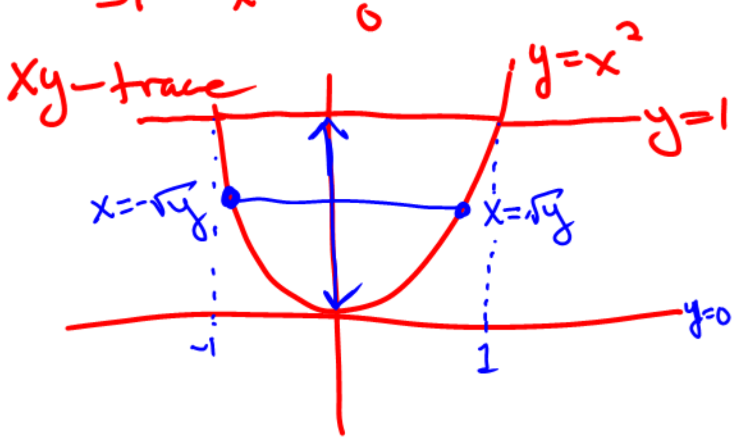
C. $\int\limits_{0}^{1}\int\limits_{0}^{1}\int\limits_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\ dxdydz$ [makes y independent of z]

D. $\int\limits_{0}^{1}\int\limits_{0}^{1-z}\int\limits_{-1}^{-\sqrt{y}} f(x,y,z)\ dxdydz + \int\limits_{0}^{1}\int\limits_{0}^{1-z}\int\limits_{\sqrt{y}}^{1} f(x,y,z)\ dxdydz$

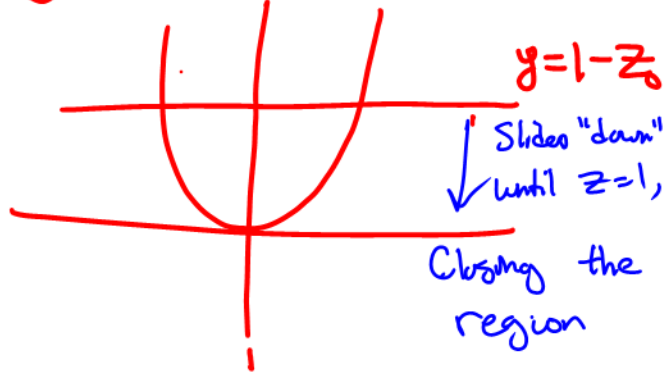
[assumes the projection is $-1 \le x \le 1, 0 \le y \le x^2$]

E. $\int\limits_{0}^{1}\int\limits_{0}^{1-z}\int\limits_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\ dxdydz$

$$\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} f\ dz\,dy\,dx \left\}\begin{array}{l} -\sqrt{y} \le x \le \sqrt{y} \\ 0 \le y \le 1-z \\ 0 \le z \le 1 \end{array}\right.$$



Cross Section at $z = z_0$



3. [8 points] Consider the curve C in $\mathbb{R}^2$ defined by $y - x^3 = 0$. Find a constant k for which the vector field $\langle -x^2, k\rangle$ is orthogonal to C at **every** point along C.

   A. $k = -\frac{1}{3}$

   B. $k = 0$

   C. $k = \frac{1}{3}$

   D. $k = 3$

   E. Such a k does not exist

This is a lot easier if you parametrize C and find k s.t. $\vec{F} \cdot \vec{r}\,' \equiv 0$, but I'll do it the way the textbook intends you to do this.

Let C be $g \equiv 0$ where $g(x,y) = y - x^3$. $\vec{F} \perp C$ when $\vec{F}$ is parallel to ∇g. Ie, there is a λ such that

$$\vec{F} = \lambda \nabla g$$

$$\langle -x^2, k\rangle = \lambda \langle -3x^2, 1\rangle$$

If $x=0$, any λ works. Assume $x \neq 0$

$$\langle -x^2, k\rangle = \langle -3\lambda x^2, \lambda\rangle$$

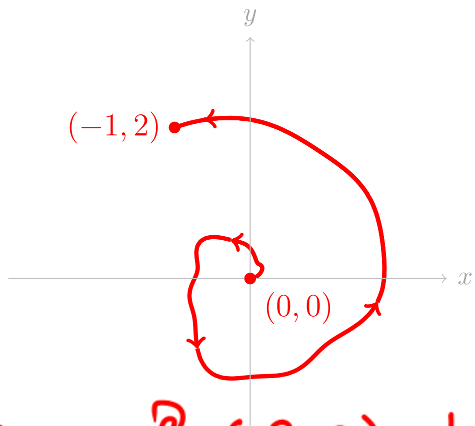
$$\Rightarrow \begin{cases} -x^2 = -3\lambda x^2 \\ \lambda = k \end{cases} \Rightarrow -x^2 = -3kx^2$$

$$x^2(3k-1) = 0 \quad ; x \neq 0$$

$$3k-1 = 0$$

$$k = \frac{1}{3}$$

4. [8 points] For the vector field $\mathbf{F}(x,y) = (2xy-1)\mathbf{i} + (x^2+2)\mathbf{j}$, find $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the noisy spiral depicted below starting at the origin and ending at $(-1,2)$.



$\frac{\partial}{\partial y}(2xy-1) = 2x = \frac{\partial}{\partial x}(x^2+2)$, hence $\vec{F}$ is conservative.

We look for a potential function φ.

$$\varphi(x,y) = \int (2xy-1)\,\partial x = x^2y - x + g(y)$$

$$\varphi(x,y) = \int (x^2+2)\,\partial y = x^2y + 2y + f(x)$$

for f a fcn of x and g a fcn of y.

Find $f(x)$ and $g(y)$ so quantities are equal. Set $g(y) = 2y$ and $f(x) = -x$ to get

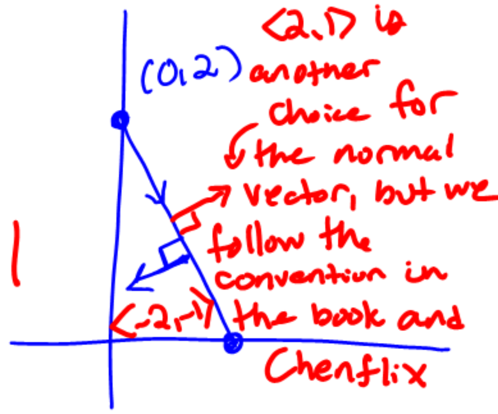
$$\varphi(x,y) = x^2y - x + 2y$$

Now, by the Fundamental Theorem of Line Integrals

$$\int_C \vec{F}\cdot d\vec{r} = \varphi(-1,2) - \varphi(0,0)$$
$$= (-1)^2(2) - (-1) + 2(2) - 0$$
$$= 7$$

$\int_C \mathbf{F}\cdot d\mathbf{r} =$ 7

5. [8 points] Let $\mathbf{F}(x,y) = \langle y, x \rangle$ and let C be the line segment from $(0,2)$ to $(1,0)$. Find the flux across C.



Parametrize the line segment

$$\vec{r}(t) = (1-t)\langle 0,2\rangle + t\langle 1,0\rangle$$
$$= \langle t, 2-2t\rangle \ ; \ 0 \le t \le 1$$

$$\vec{r}\,'(t) = \langle 1, -2\rangle$$

We'll find the normal vector the formal way

$$\vec{r}\,'(t) \times \vec{k} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -2\vec{i} - \vec{j} = \langle -2, -1\rangle$$

$$\hat{n} = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} \times \vec{k} = \frac{\vec{r}\,'(t) \times \vec{k}}{|\vec{r}\,'(t)|} = \frac{1}{|\vec{r}\,'(t)|}\langle -2, -1\rangle$$

$$\vec{F}(\vec{r}(t)) = \langle y(t), x(t)\rangle = \langle 2-2t, t\rangle$$

$$\text{Flux} = \int_C \vec{F}\cdot\hat{n}\,ds = \int_0^1 \left[\vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}\,'(t)\times\vec{k}}{|\vec{r}\,'(t)|}\right] |\vec{r}\,'(t)|\,dt$$

$$= \int_0^1 \vec{F}(\vec{r}(t)) \cdot (\vec{r}\,'(t) \times \vec{k})\,dt = \int_0^1 \langle 2-2t, t\rangle \cdot \langle -2, -1\rangle\,dt$$

$$= -\int_0^1 (4-4t+t) = -\int_0^1 (4-3t)\,dt = -\left[4t - \frac{3}{2}t^2\right]_0^1 = -\frac{5}{2}$$

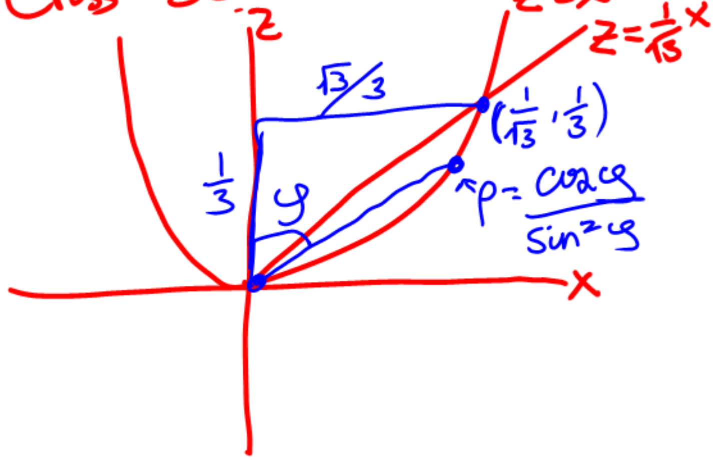
Flux= $-\frac{5}{2}$

6. [8 points] Let E be the region inside the paraboloid $z = x^2 + y^2$ outside of the cone $z = \frac{1}{\sqrt{3}}\sqrt{x^2+y^2}$. Use spherical coordinates to set up an integral for the volume of E.

Simplify your answer, but **do NOT evaluate the integral.**

(It may be helpful to sketch a 2-dimensional cross section of E.)

Cross Section



Find bounds for ρ:

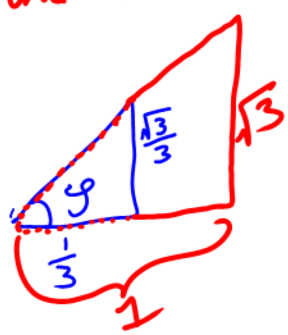
$$z = x^2+y^2$$

$$\rho\cos\varphi = (\rho\sin\varphi\cos\theta)^2+(\rho\sin\varphi\sin\theta)^2$$

$$\rho\cos\varphi = \rho^2\sin^2\varphi$$

$$\rho = \frac{\cos\varphi}{\sin^2\varphi}$$

Find Bounds for φ:



φ starts at $\tan^{-1}\left(\frac{\sqrt{3}/3}{1/3}\right) = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$ and ends after a quarter turn ($\frac{\pi}{2}$ radians).

For θ, clearly $0 \leq \theta \leq 2\pi$

So, Volume $= \int_E dV = \int_{\theta=0}^{2\pi}\int_{\varphi=\frac{\pi}{3}}^{\pi/2}\int_{\rho=0}^{\frac{\cos\varphi}{\sin^2\varphi}} \rho^2\sin\varphi \, d\rho \, d\varphi \, d\theta$

Volume= $\int_0^{2\pi}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\int_0^{\cos\phi/\sin^2\phi} \rho^2\sin\phi \; d\rho d\phi d\theta$

7. [8 points] A spring made of a thin wire is twisted into the helix:

$$\vec{r}(t) = \langle 2\cos t, 2\sin t, t^2 \rangle; \quad 0 \le t \le 6\pi$$

Find the mass of the spring if the density at the point (x, y, z) is $\rho(x, y, z) = \sqrt{z}$.

$$\vec{r}(t) = \langle 2\cos t, 2\sin t, t^2 \rangle$$

$$\vec{r}'(t) = \langle -2\sin t, 2\cos t, 2t \rangle = 2\langle -\sin t, \cos t, t \rangle$$

$$ds = |\vec{r}'(t)|\, dt = 2\sqrt{1+t^2}\, dt$$

$$\text{Mass} = \int_{\text{Helix}} \rho\, ds = \int_{\text{Helix}} \sqrt{z}\, ds = \int_0^{6\pi} \sqrt{t^2}(2)\sqrt{1+t^2}\, dt$$

$$= \int_0^{6\pi} \sqrt{1+t^2}\,(2t)\, dt = \frac{2}{3}(1+t^2)^{\frac{3}{2}}\Big]_0^{6\pi} = \frac{2}{3}\left[(1+36\pi^2)^{\frac{3}{2}} - 1\right]$$

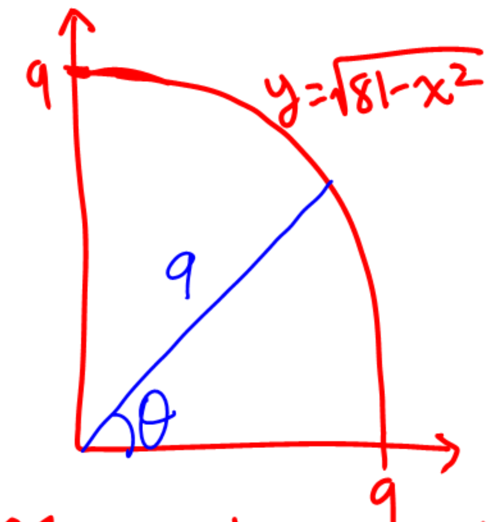
Mass= $\frac{2}{3}\left[(1+36\pi^2)^{3/2} - 1\right]$

8. [8 points] Set up an integral equivalent to the one below using cylindrical coordinates.

   Simplify your answer, but **do NOT evaluate the integral.**

$$\int_0^9 \int_0^{\sqrt{81-x^2}} \int_0^{\sqrt{x^2+y^2}} xyz \; dzdydx$$

Determine bounds for r and θ



$0 \le r \le 9$

$0 \le \theta \le \frac{\pi}{2}$

For z,

$0 \le z \le \sqrt{x^2+y^2}$

$0 \le z \le r$

Thus, the integral is

$$\int_{\theta=0}^{\pi/2} \int_{r=0}^{9} \int_{z=0}^{r} (r\cos\theta)(r\sin\theta)\, z \, dz \, \underline{\underline{(r)}} dr \, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{9} \int_{z=0}^{r} r^3 z \sin\theta \cos\theta \, dz \, dr \, d\theta$$

Integral: $\int_0^{\pi/2} \int_0^9 \int_0^r (r\cos\theta)(r\sin\theta) z dz (r) dr d\theta = \int_0^{\pi/2} \int_0^9 \int_0^r r^3 z \cos\theta \sin\theta \; dzdrd\theta$

9. [8 points] A loaded die $D = \{(x, y, z) | \ 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$ has a density function $\rho(x, y, z) = 1 - z$. If the mass is $\frac{1}{2}$, what is the $z-$coordinate of the die's center of mass?

Mass $= \int_{Die} \rho \, dV$, but it is given to us that it is $\frac{1}{2}$. So the z-coordinate is

$$\bar{z} = \frac{1}{Mass} \int_{Die} z \rho \, dV = \frac{1}{\frac{1}{2}} \int_0^1 \int_0^1 \int_0^1 z(1-z) \, dx \, dy \, dz$$

$$= 2 \int_0^1 [z - z^2] \, dz = 2\left[\frac{1}{2}z^2 - \frac{1}{3}z^3\right]_0^1 = 2\left[\frac{1}{2} - \frac{1}{3}\right]$$

$$= 2\left[\frac{1}{6}\right] = \frac{1}{3}$$

$z = \frac{1}{3}$

10. [8 points] A point charge placed at the origin produces an electric field; the force acting on a particle at (x, y, z) by the field is

$$\mathbf{F}(x,y,z) = \frac{-1}{(x^2+y^2+z^2)^{3/2}}\langle x,y,z\rangle; \quad x^2+y^2+z^2 \neq 0$$

If a particle moves along the curve defined by $\vec{r}(t) = \langle 1, 2t, 3t\rangle$ $(0 \le t \le 1)$, set up an integral that computes the work done by the force field on the particle.

Simplify your answer, but **do NOT evaluate the integral.**

$\vec{r}(t) = \langle 1, 2t, 3t\rangle$; $0 \le t \le 1$

$\vec{r}\,'(t) = \langle 0, 2, 3\rangle$

$$\vec{F}(\vec{r}(t)) = \frac{-1}{(1^2+(2t)^2+(3t)^2)^{3/2}}\langle 1, 2t, 3t\rangle$$

$$\text{Work} = \int_0^1 \frac{-1}{(1+13t^2)^{3/2}}\langle 1, 2t, 3t\rangle \cdot \langle 0,2,3\rangle\, dt$$

$$= \int_0^1 \frac{-13t}{(1+13t^2)^{\frac{3}{2}}}\, dt$$

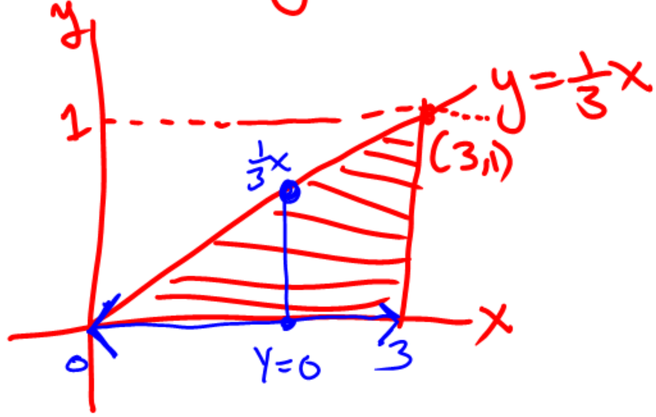
Can be integrated with a u-sub, but we didn't ask you to do that on the exam.

Work $= \int_0^1 \frac{-13t}{(1+13t^2)^{3/2}}\, dt$

11. [8 points] Reverse the order of integration and evaluate:

$$\int_0^1 \int_{3y}^3 e^{x^2}\ dxdy$$

Draw Region of Integration $3y \le x \le 3$, $0 \le y \le 1$



$$\int_0^1 \int_{3y}^3 e^{x^2} dxdy = \int_{x=0}^3 \int_{y=0}^{\frac{1}{3}x} e^{x^2} dy\, dx$$

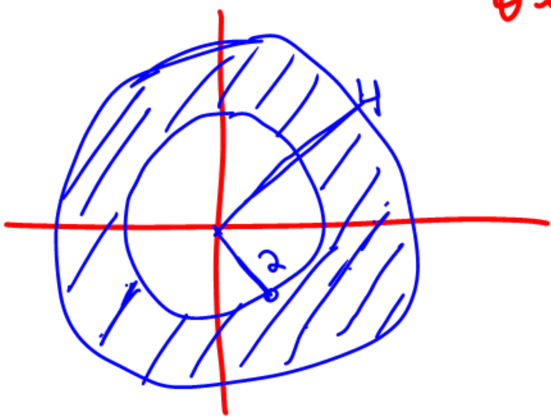
$$= \int_0^3 \frac{1}{3} x e^{x^2} dx = \frac{1}{3} \cdot \frac{1}{2} \int_0^3 e^{x^2} (2x) dx = \frac{1}{6} \left[e^{x^2} \right]_0^3$$

$$= \frac{1}{6} \left[e^9 - 1 \right]$$

$\int_0^1 \int_{3y}^3 e^{x^2}\ dxdy =$ $\frac{1}{6}(e^9 - 1)$

12. [8 points] Given a point (r, θ) in polar coordinates, let f be defined by $f(r, \theta) = \frac{1}{r^2}$. Find the average value of $f(r, \theta)$ over the annulus $\{(r, \theta)|\ 2 \le r \le 4\}$.

Area of Annulus: $\int_{\theta=0}^{2\pi} \int_{r=2}^{4} r\,dr\,d\theta = \pi(4)^2 - \pi(2)^2 = 12\pi$



Average:

$$\bar{f} = \frac{1}{\text{Area}} \int_{\text{Annulus}} \frac{1}{r^2}\,dA = \frac{1}{12\pi} \int_{\theta=0}^{2\pi} \int_{r=2}^{4} \frac{1}{r^2}\, r\,dr\,d\theta$$

$$= \frac{2\pi}{12\pi} \int_2^4 \frac{1}{r}\,dr = \frac{1}{6} \ln|r| \Big]_2^4$$

$$= \frac{1}{6}[\ln 4 - \ln 2]$$

$$= \frac{1}{6} \ln 2$$

Answer: $\frac{1}{6} \ln 2$